The integral cohomology groups of configuration spaces of pairs of points in real projective spaces

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Abstract

We compute the integral homology and cohomology groups of configuration spaces of two distinct points on a given real projective space. The explicit answer is related to the (known multiplicative structure in the) integral cohomology—with simple and twisted coefficients—of the dihedral group of order 8 (in the case of unordered configurations) and the elementary abelian 2-group of rank 2 (in the case of ordered configurations). As an application, we complete the computation of the symmetric topological complexity of real projective spaces $\mathrm{P}^{2^i+\delta}$ with $i \geq 0$ and $0 \leq \delta \leq 2$.

Key words and phrases: *2-point configurations of real projective spaces; dihedral group of order* 8*; twisted Poincaré duality and torsion linking form; symmetric topological complexity; Bockstein, Cartan-Leray, and Serre spectral sequences.*
2010 Mathematics Subject Classification: Primary: 55R80, 55T10; Secondary: 55M30, 57R19, 57R40.

1 Introduction and description of main results

Unless explicitly indicated otherwise, the notation $H^*(X)$ refers to integral cohomology groups of a space X where a simple system of local coefficients is used. The cyclic group with 2^e elements is denoted by $\mathbb{Z}_{2^e}$. In the case $e = 1$ we also use the notation $\mathbb{F}_2$ if the field structure is to be noted. It will be convenient to use the notation $\langle k \rangle$ for the elementary abelian 2-group of rank k, and write $\{k\}$ as a shorthand for $\langle k \rangle \oplus \mathbb{Z}_4$.

We address the problem of computing the integral homology and cohomology groups of the configuration spaces $F(\mathrm{P}^m, 2)$ and $B(\mathrm{P}^m, 2)$ of two distinct points, ordered and unordered respectively, in the m-dimensional real projective space P^m. Our main results are presented in Theorems 1.1, 1.2, 1.9, and 1.10. The first two of these take the following explicit form:

*Partially supported by CONACYT Research Grant 102783

Theorem 1.1. *For $n > 0$,*

$$H^i(F(\mathrm{P}^{2n},2)) = \begin{cases} \mathbb{Z}, & i=0 \quad or \quad i=4n-1; \\ \left\langle \frac{i}{2}+1 \right\rangle, & i \quad even, \;\; 1 \le i \le 2n; \\ \left\langle \frac{i-1}{2} \right\rangle, & i \quad odd, \;\; 1 \le i \le 2n; \\ \left\langle 2n+1-\frac{i}{2} \right\rangle, & i \quad even, \;\; 2n < i < 4n-1; \\ \left\langle 2n-\frac{i+1}{2} \right\rangle, & i \quad odd, \;\; 2n < i < 4n-1; \\ 0, & otherwise. \end{cases}$$

For $n \ge 0$,

$$H^i(F(\mathrm{P}^{2n+1},2)) = \begin{cases} \mathbb{Z}, & i=0; \\ \left\langle \frac{i}{2}+1 \right\rangle, & i \quad even, \;\; 1 \le i \le 2n; \\ \left\langle \frac{i-1}{2} \right\rangle, & i \quad odd, \;\; 1 \le i \le 2n; \\ \mathbb{Z} \oplus \langle n \rangle, & i = 2n+1; \\ \left\langle 2n+1-\frac{i}{2} \right\rangle, & i \quad even, \;\; 2n+1 < i \le 4n+1; \\ \left\langle 2n+1-\frac{i-1}{2} \right\rangle, & i \quad odd, \;\; 2n+1 < i \le 4n+1; \\ 0, & otherwise. \end{cases}$$

Theorem 1.2. *Let $0 \le b \le 3$. For $n > 0$,*

$$H^{4a+b}(B(\mathrm{P}^{2n},2)) = \begin{cases} \mathbb{Z}, & 4a+b=0 \quad or \quad 4a+b=4n-1; \\ \{2a\}, & b=0<a, \;\; 4a+b \le 2n; \\ \langle 2a \rangle, & b=1, \;\; 4a+b \le 2n; \\ \langle 2a+2 \rangle, & b=2, \;\; 4a+b \le 2n; \\ \langle 2a+1 \rangle, & b=3, \;\; 4a+b \le 2n; \\ \{2n-2a\}, & b=0, \;\; 2n < 4a+b < 4n-1; \\ \langle 2n-2a-1 \rangle, & b=1, \;\; 2n < 4a+b < 4n-1; \\ \langle 2n-2a \rangle, & b=2, \;\; 2n < 4a+b < 4n-1; \\ \langle 2n-2a-2 \rangle, & b=3, \;\; 2n < 4a+b < 4n-1; \\ 0, & otherwise. \end{cases}$$

For $n \geq 0$,

$$H^{4a+b}(B(\mathrm{P}^{2n+1},2)) = \begin{cases} \mathbb{Z}, & 4a+b=0; \\ \{2a\}, & b=0<a,\ \ 4a+b<2n+1; \\ \langle 2a\rangle\,, & b=1,\ \ 4a+b<2n+1; \\ \langle 2a+2\rangle\,, & b=2,\ \ 4a+b<2n+1; \\ \langle 2a+1\rangle\,, & b=3,\ \ 4a+b<2n+1; \\ \mathbb{Z}\oplus\langle n\rangle, & 4a+b=2n+1; \\ \{2n-2a\}, & b=0,\ \ 2n+1<4a+b\leq 4n+1; \\ \langle 2n+1-2a\rangle, & b=1,\ \ 2n+1<4a+b\leq 4n+1; \\ \langle 2n-2a\rangle, & b\in\{2,3\},\ \ 2n+1<4a+b\leq 4n+1; \\ 0, & \textit{otherwise.} \end{cases}$$

Theorems 1.1 and 1.2 can be coupled with the Universal Coefficient Theorem (UCT), expressing homology in terms of cohomology (e.g. [34, Theorem 56.1]), in order to give explicit descriptions of the corresponding integral homology groups. Another immediate consequence is that, together with Poincaré duality (in its not necessarily orientable version, cf. [24, Theorem 3H.6] or [36, Theorem 4.51]), Theorems 1.1 and 1.2 give a corresponding explicit description of the w_1-twisted homology and cohomology groups of $F(\mathrm{P}^m,2)$ and $B(\mathrm{P}^m,2)$. Details are given in Section 4.

It is to be observed that Theorem 1.2 fully extends cohomological calculations for $B(\mathrm{P}^m,2)$ in [3] (using a different approach). Rather elaborate Bockstein spectral sequence considerations in that paper led Bausum to a description of a few of the cohomology groups in Theorem 1.2—groups that are close to the top cohomological dimension $2m-1$. In turn, this leads to a description of sets $\mathrm{Emb}_e(\mathrm{P}^m)$ of isotopy classes of smooth embeddings of P^m in $\mathbb{R}^{2m-e}$ for low values of e (as low as $e \leq 2$). Similar results were obtained by Larmore and Rigdon (note the implicit hypothesis $m>3$ in [29, Section 4])[1]. More recently, Section 3 in [17] explains how results like Theorems 1.1 and 1.2 could potentially lead to new embedding-type information about projective spaces (Theorem 1.4 and Remark 1.5 below are based on such a viewpoint).

Theorem 1.1 implies that the torsion in $H^*(F(\mathrm{P}^m,2))$ is annihilated by 2. This observation and a standard argument using the transfer of the double cover $F(\mathrm{P}^m,2)\to B(\mathrm{P}^m,2)$ show[2] that the torsion subgroup of $H^*(B(\mathrm{P}^m,2))$ is annihilated by 4. Theorem 1.2 then shows this is a sharp bound, as $H^*(B(\mathrm{P}^m,2))$ has 4-torsion elements in dimensions 4ℓ for $0<\ell<m/2$. Thus, Theorem 1.2 proves a recent conjecture of Fred Cohen claiming that $H^*(B(\mathrm{P}^m,2))$ has 2-torsion (for $m>1$) and 4-torsion (for $m>2$), but no 8-torsion.

[1]We thank Sadok Kallel for pointing out the results in [3] and [29].
[2]We thank Fred Cohen for pointing out his argument.

Remark 1.3. Note that, after inverting 2, both $B(\mathrm{P}^m, 2)$ and $F(\mathrm{P}^m, 2)$ are homology spheres. This assertion can be considered as a partial generalization of the fact that both $F(\mathrm{P}^1, 2)$ and $B(\mathrm{P}^1, 2)$ have the homotopy type of a circle; for $B(\mathrm{P}^1, 2)$ this follows from Lemma 1.8 and Example 3.4 below, while the situation for $F(\mathrm{P}^1, 2)$ comes from the fact that P^1 is a Lie group—so that $F(\mathrm{P}^1, 2)$ is in fact diffeomorphic to $S^1 \times (S^1 - \{1\})$. In particular, any product of positive dimensional classes in either $H^*(F(\mathrm{P}^1, 2))$ or $H^*(B(\mathrm{P}^1, 2))$ is trivial. The trivial-product property also holds for both $H^*(F(\mathrm{P}^2, 2))$ and $H^*(B(\mathrm{P}^2, 2))$ in view of the P^2-case in Theorems 1.1 and 1.2. But for $m \geq 3$ both $F(\mathrm{P}^m, 2)$ and $B(\mathrm{P}^m, 2)$ should have useful integral cohomology rings ([17] contains partial information on the case $m = 3$, as well as an application along the lines of Theorem 1.4 below), and this motivates the considerations in the rest of this introductory section.

The results in this paper go a bit further than a plain computation of cohomology groups. Our ultimate motivation comes from the possibility of deducing new information on the Euclidean embedding dimension of projective spaces based on a good hold of the cohomology rings of the relevant configuration spaces. Explicitly, let

$$B(\mathrm{P}^m, 2) \xrightarrow{u} \mathrm{P}^\infty \tag{1}$$

classify the obvious double cover $F(\mathrm{P}^m, 2) \to B(\mathrm{P}^m, 2)$. Then, with the seven possible exceptions[3] of m explicitly described in [17, Equation (8)], $\mathrm{Emb}(\mathrm{P}^m)$—the dimension of the smallest Euclidean space in which P^m can be smoothly embedded—is characterized as the smallest integer $n = n(m)$ such that the map in (1) can be homotopy compressed into P^{n-1}. Furthermore, the main result in [18] asserts that, without restriction on m, the number $n(m)$ agrees with Farber's symmetric topological complexity[4] of P^m, $\mathrm{TC}^S(\mathrm{P}^m)$, an invariant based on the motion planning problem in robotics. From such a viewpoint, a proper understanding of the multiplicative height of $u^*(z)$, where z is the generator in $H^2(\mathrm{P}^\infty)$, gives lower bounds on the values that n can take—potentially leading to new information on the embedding problem of real projective spaces. The idea actually goes back at least as far as [21], where mod 2 coefficients (and obstruction theory) are used. But the $\mathbb{Z}_4$ groups appearing in Theorem 1.2 seem to carry finer information not yet explored[5]. For instance, the strategy using integral coefficients has recently been exploited in [17] in order to compute $\mathrm{TC}^S(\mathrm{SO}(3))$—identifying it as the unique obstruction in Goodwillie's embedding Taylor tower for P^3.

As an application of the cohomological results in this paper, our next result completes the computation started in [18] of the symmetric topological complexity of projective spaces of the form $\mathrm{P}^{2^i+\delta}$ in the range $i \geq 0$ and $0 \leq \delta \leq 2$.

[3] Remark 1.5 below observes that we can now rule out the first of these potential exceptions.

[4] As indicated in Definition 8.1 at the end of the paper, here we use the reduced version of Farber's TC^S, i.e. we choose to normalize the Schwarz genus of a product fibration $F \times B \to B$ to be 0—not 1.

[5] Compare with the situation in [4] where the topological Borsuk problem for $\mathbb{R}^3$ is studied via Fadell-Husseini index theory.

Theorem 1.4. $\mathrm{TC}^S(\mathrm{P}^5) = \mathrm{TC}^S(\mathrm{P}^6) = 9$.

Section 8 starts with a discussion exhibiting the case of P^6 as giving the unique exceptional numerical value for $\mathrm{TC}^S(\mathrm{P}^{2^i+\delta})$ in the range $i \geq 0$ and $0 \leq \delta \leq 2$.

Remark 1.5. Since $\mathrm{Emb}(\mathrm{P}^5) = 9$ ([25, 30]), the list in [17] of seven exceptional values of m for which the equality $\mathrm{Emb}(\mathrm{P}^m) = \mathrm{TC}^S(\mathrm{P}^m)$ *could* fail reduces now to $\{6, 7, 11, 12, 14, 15\}$. Note that 6 is the smallest m for which $\mathrm{Emb}(\mathrm{P}^m)$ is unknown: $\mathrm{Emb}(\mathrm{P}^6) \in \{9, 10, 11\}$ is the best assertion known to date ([8, 30]). On the other hand, Theorem 1.4 obviously implies $\mathrm{TC}^S(\mathrm{P}^7) \geq 9$, improving by 1 the previously known best lower bound for $\mathrm{TC}^S(\mathrm{P}^7)$ noted in [17, Table 1]. In fact, taking into account Rees' PL embedding $\mathrm{P}^7 \subset \mathbb{R}^{10}$ constructed in [38], the above considerations imply that both $\mathrm{TC}^S(\mathrm{P}^7)$ and $\mathrm{Emb}_{\mathrm{PL}}(\mathrm{P}^7)$ lie in $\{9, 10\}$, which contrasts with the best known assertion about the embedding dimension of P^7, namely $\mathrm{Emb}(\mathrm{P}^7) \in \{9, 10, 11, 12\}$ ([23, 31]). Despite the fact that the equality $\mathrm{Imm}(\mathrm{P}^m) = \mathrm{TC}(\mathrm{P}^m)$ actually has three exceptions (related to the Hopf invariant one problem), the above observations lead us to think that the equality $\mathrm{Emb}(\mathrm{P}^m) = \mathrm{TC}^S(\mathrm{P}^m)$ should actually hold for every m, at least if Emb is interpreted as *topological* embedding dimension. From such a perspective, it would be highly desirable to know whether P^6 topologically embeds in $\mathbb{R}^9$. On the other hand, it does not seem likely that P^7 could possibly embed in $\mathbb{R}^9$ (even topologically), and the techniques proving Theorem 1.4 (using perhaps a cohomology theory better suited than singular cohomology) might allow us to formalize our intuition—we hope to come back to such a point elsewhere.

A profitable approach to the kind of applications in the previous paragraphs comes from using Handel's observation that (1) factors through the classifying space of the dihedral group D_8. Namely, (1) is homotopic to the composite

$$B(\mathrm{P}^m, 2) \xrightarrow{p} BD_8 \xrightarrow{q} \mathrm{P}^\infty \tag{2}$$

where p is specified in Notation 1.7 below, and q is specified in Remark 2.10 at the end of Section 2. Now, not only are $H^*(\mathrm{P}^\infty)$ and $H^*(BD_8)$ well-known rings, but the induced ring map q^* is well understood (Remark 2.10). But most importantly, the induced map p^* surjects onto the torsion subgroups of $H^*(B(\mathrm{P}^m, 2))$ except, perhaps, for $m \equiv 3 \bmod 4$ (Theorems 1.9 and 1.10 below). So, an eventual study of the multiplicative height of $u^*(z)$, and of the ring $H^*(B(\mathrm{P}^m, 2))$ for that matter, can be reduced to having a good hold on the kernel of p^*, i.e., Fadell-Husseini's ideal-valued $\mathbb{Z}$-index of the D_8-restricted action of $\mathrm{O}(2)$ on $V_{m+1,2}$—see Definition 1.6, Lemma 1.8, and the considerations around (5). For the remainder of this section our attention focuses on the likely surjectivity property of p^* and, with this in mind, the following considerations (see for instance [21, §2]) are our main tool:

Definition 1.6. Recall that D_8 can be expressed as the usual wreath product extension

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to D_8 \to \mathbb{Z}_2 \to 1. \tag{3}$$

Let $\rho_1, \rho_2 \in D_8$ generate the normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, and let (the class of) $\rho \in D_8$ generate the quotient group $\mathbb{Z}_2$ so that, via conjugation, ρ switches ρ_1 and ρ_2. D_8 acts freely on the Stiefel manifold $V_{n,2}$ of orthonormal 2-frames in $\mathbb{R}^n$ by setting

$$\rho(v_1, v_2) = (v_2, v_1), \quad \rho_1(v_1, v_2) = (-v_1, v_2), \quad \text{and} \quad \rho_2(v_1, v_2) = (v_1, -v_2).$$

This describes a group inclusion $D_8 \hookrightarrow \mathrm{O}(2)$ where the rotation $\rho\rho_1$ is a generator for $\mathbb{Z}_4 = D_8 \cap \mathrm{SO}(2)$.

Notation 1.7. Throughout the paper the letter G stands for either D_8 or its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (3). Likewise, $E_m = E_{m,G}$ denotes the orbit space of the G-action on $V_{m+1,2}$ indicated in Definition 1.6, and $\theta\colon V_{m+1,2} \to E_{m,G}$ represents the canonical projection. As explained in the paragraph containing (2), our interest lies in the (kernel of the) morphism induced in cohomology by the map

$$p = p_{m,G}\colon E_m \to BG \tag{4}$$

that classifies the G-action on $V_{m+1,2}$.

Lemma 1.8 ([21, Proposition 2.6])**.** *E_m is a strong deformation retract of $B(\mathrm{P}^m, 2)$ if $G = D_8$, and of $F(\mathrm{P}^m, 2)$ if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.* □

Thus, the cohomology properties of the configuration spaces we are interested in—and of (4), for that matter—can be approached via the Cartan-Leray spectral sequence (CLSS) of the G-action on $V_{m+1,2}$. Such an analysis yields:

Theorem 1.9. *Let m be even. The map $p^*\colon H^i(BG) \to H^i(E_m)$ is:*

1. *an isomorphism for $i \leq m$;*
2. *an epimorphism with nonzero kernel for $m < i < 2m - 1$;*
3. *the zero map for $2m - 1 \leq i$.*

Theorem 1.10. *Let m be odd. The map $p^*\colon H^i(BG) \to H^i(E_m)$ is:*

1. *an isomorphism for $i < m$;*
2. *a monomorphism onto the torsion subgroup of $H^i(E_m)$ for $i = m$;*
3. *the zero map for $2m - 1 < i$.*

Further, p^* is an epimorphism with nonzero kernel for $m < i \leq 2m - 1$ except perhaps when $G = D_8$ and $m \equiv 3 \bmod 4$.

Since the ring $H^*(BG)$ is well known (see Theorem 2.3 and the comments following Lemma 2.8), the multiplicative structure of $H^*(E_m)$ through dimensions at most m follows from the four theorems stated in this section. Furthermore, much of the ring structure in larger dimensions now depends on giving explicit generators for the ideal $\mathrm{Ker}(p^*)$. In this direction we prove the following result (noticed independently by Fred Cohen using different methods):

Proposition 1.11. *Let $G = D_8$. Assume $m \not\equiv 3 \bmod 4$ and consider the map in (4). In dimensions at most $2m-1$, every nonzero element in $\mathrm{Ker}(p^*)$ has order 2, i.e. $2 \cdot \mathrm{Ker}(p^*) = 0$ in those dimensions. In fact, every 4ℓ-dimensional integral cohomology class in BD_8 generating a $\mathbb{Z}_4$-group maps under p^* into a class which also generates a $\mathbb{Z}_4$-group provided $\ell < m/2$—otherwise the class maps trivially for dimensional reasons.*

Remark 1.12. By Lemma 2.8 below, $\mathrm{Ker}(p^*)$ is also killed by multiplication by 2 when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (any m, any dimension). Our approach allows us to explicitly describe the (dimension-wise) 2-rank of $\mathrm{Ker}(p^*)$ in the cases where we know this is an $\mathbb{F}_2$-vector space (i.e. when either $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m \not\equiv 3 \bmod 4$, see Examples 5.3 and 5.7). Unfortunately the methods used in the proofs of Proposition 1.11 and Theorems 1.9 and 1.10 break down for E_{4n+3,D_8}, and Section 6 in the preliminary version [19] of this paper discusses a few such aspects, mainly focusing attention on the case $n = 0$. We hope this paper serves as a motivation to study the case of $B(\mathrm{P}^{4n+3}, 2)$ in Theorem 1.10, as well as to get a hold on the complete ring structure of $H^*(E_m)$ or, for that matter, on the kernel of p^*—aiming, for instance, at the geometric applications sketched in the paragraph containing (1).

The spectral sequence methods in this paper are similar in spirit to those in [5] and [14]. In the latter reference, Feichtner and Ziegler describe the integral cohomology rings of *ordered* configuration spaces on spheres by means of a full analysis of the Serre spectral sequence (SSS) associated to the Fadell-Neuwirth fibration $\pi\colon F(S^k, n) \to S^k$ given by $\pi(x_1, \ldots, x_n) = x_n$ (a similar study is carried out in [15], but in the context of *ordered* orbit configuration spaces). One of the main achievements of the present paper is a successful calculation of cohomology groups of *unordered* configuration spaces (on real projective spaces), where no Fadell-Neuwirth fibrations are available—instead we rely on Lemma 1.8 and the CLSS[6] of the G-action on $V_{m+1,2}$. Also worth stressing is the fact that we succeed in computing cohomology groups with *integer* coefficients, whereas the Leray spectral sequence (and its Σ_k-invariant version) for the inclusion $F(X, k) \hookrightarrow X^k$ has proved to be effectively computable mainly when *field* coefficients are used ([16, 42]).

A major obstacle we have to confront (not present in [14]) comes from the fact that the spectral sequences we encounter often have non-simple systems of local coefficients. This is also the situation in [5], where the two-hyperplane case of Grünbaum's mass partition

[6]Our CLSS calculations can also be done in terms of the SSS of the fibration $V_{m+1,2} \xrightarrow{\theta} E_{m,G} \xrightarrow{p} BG$.

problem ([20]) is studied from the Fadell-Husseini index theory viewpoint [9]. Indeed, Blagojević and Ziegler deal with twisted coefficients in their main SSS, namely the one associated to the Borel fibration

$$S^m \times S^m \to ED_8 \times_{D_8} (S^m \times S^m) \xrightarrow{\overline{p}} BD_8 \tag{5}$$

where the D_8-action on $S^m \times S^m$ is the obvious extension of that in Definition 1.6. Now, the main goal in [5] is to describe the kernel of the map induced by $\overline{p}$ in integral cohomology—the so-called Fadell-Husseini ($\mathbb{Z}$-)index of D_8 acting on $S^m \times S^m$, $\mathrm{Index}_{D_8}(S^m \times S^m)$. Since D_8 acts freely on $V_{m+1,2}$, $\mathrm{Index}_{D_8}(S^m \times S^m)$ is contained in the kernel of the map induced in integral cohomology by the map $p\colon E_m \to BD_8$ in Proposition 1.11 (whether or not $m \equiv 3 \bmod 4$). In particular, the work in [5] can be used to identify explicit elements in $\mathrm{Ker}(p^*)$ and, as observed in Remark 1.12, our approach allows us to assess, for $m \not\equiv 3 \bmod 4$ (in Examples 5.3 and 5.7), how much of the actual kernel is still lacking description: [5] gives just a bit less than half the expected elements in $\mathrm{Ker}(p^*)$.

2 Preliminary cohomology facts

As shown in [1] (see also [21] for a straightforward approach), the mod 2 cohomology of D_8 is a polynomial ring on three generators $x, x_1, x_2 \in H^*(BD_8; \mathbb{F}_2)$, the first two of dimension 1, and the last one of dimension 2, subject to the single relation $x^2 = x \cdot x_1$. The classes x_i are the restrictions of the universal Stiefel-Whitney classes w_i ($i = 1, 2$) under the map corresponding to the group inclusion $D_8 \subset \mathrm{O}(2)$ in Definition 1.6. On the other hand, the class x is not characterized by the relation $x^2 = x \cdot x_1$, but by the requirement that, for all m, x pulls back to (1) under the map p_{m,D_8} in (4)—see [21, Proposition 3.5]. In particular:

Lemma 2.1. *For $i \geq 0$, $H^i(BD_8; \mathbb{F}_2) = \langle i+1 \rangle$.* □

Corollary 2.2. *For any m,*

$$H^i(B(\mathrm{P}^m, 2); \mathbb{F}_2) = \begin{cases} \langle i+1 \rangle, & 0 \leq i \leq m-1; \\ \langle 2m - i \rangle, & m \leq i \leq 2m-1; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The assertion for $i \geq 2m$ follows from Lemma 1.8 and dimensional considerations. Poincaré duality implies that the assertion for $m \leq i \leq 2m-1$ follows from that for $0 \leq i \leq m-1$. Since $V_{m+1,2}$ is $(m-2)$-connected, the assertion for $0 \leq i \leq m-1$ follows from Lemma 2.1, using the fact (a consequence of [21, Proposition 3.6 and (3.8)]) that, in the mod 2 SSS for the fibration $V_{m+1,2} \xrightarrow{\theta} E_{m,D_8} \xrightarrow{p} BD_8$, the two indecomposable elements in $H^*(V_{m+1,2}; \mathbb{F}_2)$ transgress to nontrivial elements. □

Let $\mathbb{Z}_\alpha$ denote the $\mathbb{Z}[D_8]$-module whose underlying group is free on a generator α on which each of $\rho, \rho_1, \rho_2 \in D_8$ acts via multiplication by -1 (in particular, elements in $D_8 \cap \mathrm{SO}(2)$ act trivially). Corollaries 2.4 and 2.5 below are direct consequences of the following description, proved in [22] (see also [5, Theorem 4.5]), of the ring $H^*(BD_8)$ and of the $H^*(BD_8)$-module $H^*(BD_8; \mathbb{Z}_\alpha)$:

Theorem 2.3 (Handel [22])**.** *$H^*(BD_8)$ is generated by classes μ_2, ν_2, λ_3, and κ_4 subject to the relations $2\mu_2 = 2\nu_2 = 2\lambda_3 = 4\kappa_4 = 0$, $\nu_2^2 = \mu_2\nu_2$, and $\lambda_3^2 = \mu_2\kappa_4$. $H^*(BD_8; \mathbb{Z}_\alpha)$ is the free $H^*(BD_8)$-module on classes α_1 and α_2 subject to the relations $2\alpha_1 = 4\alpha_2 = 0$, $\lambda_3\alpha_1 = \mu_2\alpha_2$, and $\kappa_4\alpha_1 = \lambda_3\alpha_2$. Subscripts in the notation of these six generators indicate their cohomology dimensions.* □

The notation a_2, b_2, c_3, and d_4 was used in [22] instead of the current μ_2, ν_2, λ_3, and κ_4. The change is made in order to avoid confusion with the generic notation d_i for differentials in the several spectral sequences considered in this paper.

Corollary 2.4. *For $a \geq 0$ and $0 \leq b \leq 3$,*

$$H^{4a+b}(BD_8) = \begin{cases} \mathbb{Z}, & (a,b) = (0,0); \\ \{2a\}, & b = 0 < a; \\ \langle 2a \rangle, & b = 1; \\ \langle 2a+2 \rangle, & b = 2; \\ \langle 2a+1 \rangle, & b = 3. \end{cases}$$ □

Corollary 2.5. *For $a \geq 0$ and $0 \leq b \leq 3$,*

$$H^{4a+b}(BD_8; \mathbb{Z}_\alpha) = \begin{cases} \langle 2a \rangle, & b = 0; \\ \langle 2a+1 \rangle, & b = 1; \\ \{2a\}, & b = 2; \\ \langle 2a+2 \rangle, & b = 3. \end{cases}$$ □

We show that, up to a certain symmetry condition (exemplified in Table 1 at the end of Section 4), the groups explicitly described by Corollaries 2.4 and 2.5 delineate the additive structure of the graded group $H^*(B(\mathrm{P}^m, 2))$. The corresponding situation for $H^*(F(\mathrm{P}^m, 2))$ uses the following well-known analogues of Lemma 2.1 and Corollaries 2.2, 2.4 and 2.5:

Lemma 2.6. *For $i \geq 0$, $H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{F}_2) = \langle i+1 \rangle$.* □

Lemma 2.7. *For any m,*

$$H^i(F(\mathrm{P}^m, 2); \mathbb{F}_2) = \begin{cases} \langle i+1 \rangle, & 0 \leq i \leq m-1; \\ \langle 2m-i \rangle, & m \leq i \leq 2m-1; \\ 0, & \textit{otherwise}. \end{cases}$$ □

Lemma 2.8. *For $i \geq 0$,*

$$H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty) = \begin{cases} \mathbb{Z}, & i = 0; \\ \left\langle \frac{i}{2} + 1 \right\rangle, & i \text{ even }, i > 0; \\ \left\langle \frac{i-1}{2} \right\rangle, & \text{otherwise.} \end{cases}$$

$$H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{Z}_\alpha) = \begin{cases} \left\langle \frac{i}{2} \right\rangle, & i \text{ even}; \\ \left\langle \frac{i+1}{2} \right\rangle, & i \text{ odd.} \end{cases}$$

Here $\mathbb{Z}_\alpha$ is regarded as a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-module via the restricted structure coming from the inclusion $\mathbb{Z}_2 \times \mathbb{Z}_2 \hookrightarrow D_8$. □

Here are some brief comments on the proofs of Lemmas 2.6–2.8. Of course, the ring structure $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{F}_2) = \mathbb{F}_2[x_1, y_1]$ is standard (as in Theorem 2.3, subscripts for the cohomology classes in this paragraph indicate dimension). On the other hand, it is easily shown (see for instance [24, Example 3E.5 on pages 306–307]) that $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty)$ is the polynomial ring over the integers on three classes x_2, y_2, and z_3 subject to the four relations

$$2x_2 = 0, \quad 2y_2 = 0, \quad 2z_3 = 0, \text{ and } \; z_3^2 = x_2 y_2 (x_2 + y_2). \tag{6}$$

These two facts yield Lemma 2.6 and the first equality in Lemma 2.8. Lemma 2.7 can be proved with the argument given for Corollary 2.2—replacing D_8 by its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (3). Finally, both equalities in Lemma 2.8 can be obtained as immediate consequences of the Künneth exact sequence (for the second equality, note that $\mathbb{Z}_\alpha$ arises as the tensor square of the standard twisted coefficients for a single factor P^∞).

Remark 2.9. For future reference we recall (again from Hatcher's book) that the mod 2 reduction map $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty) \to H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{F}_2)$, a monomorphism in positive dimensions, is characterized by $x_2 \mapsto x_1^2$, $y_2 \mapsto y_1^2$, and $z_3 \mapsto x_1 y_1 (x_1 + y_1)$.

Remark 2.10. Here are the promised details about the factorization of (1) through BD_8. We already noticed that the claimed factorization (2) is proved in [21]—for $m \geq 3$, but the restriction can be removed by naturality—where $q \colon BD_8 \to \mathrm{P}^\infty$ corresponds to the class $x \in H^1(BD_8; \mathbb{F}_2)$ at the beginning of the section. On the other hand, the extension (3) defines a fibration

$$\mathrm{P}^\infty \times \mathrm{P}^\infty \xrightarrow{\iota} BD_8 \xrightarrow{q'} \mathrm{P}^\infty,$$

and Handel's proof of [21, Proposition 3.5] characterizes x as the only nonzero element in $H^1(BD_8; \mathbb{F}_2)$ mapping trivially under the fiber inclusion ι. Thus, in fact $q = q'$. In particular, the map induced by q in integral cohomology can be computed in purely algebraic terms, using the projection in (3). Actually, since $H^*(\mathrm{P}^\infty) = \mathbb{Z}[z]/(2z)$ with $z \in H^2(\mathrm{P}^\infty)$, q^* is determined by its value on z. As the reader can easily verify, a simple exercise using the well-known resolution of the (trivial) D_8-module $\mathbb{Z}$ (see for instance [22]) shows that generators in Theorem 2.3 can be chosen so that $q^*(z) = \nu_2$.

3 Orientability properties of some quotients of $V_{n,2}$

Proofs in this section will be postponed until all relevant results have been presented. Recall that all Stiefel manifolds $V_{n,2}$ are orientable (actually parallelizable, cf. [40]). Even if some of the elements of a given subgroup H of $\mathrm{O}(2)$ fail to act on $V_{n,2}$ in an orientation-preserving way, we could still use the possible orientability of the quotients $V_{n,2}/H$ as an indication of the extent to which H, as a whole, is compatible with the orientability of the several $V_{n,2}$. For example, while every element of $\mathrm{SO}(2)$ gives an orientation-preserving diffeomorphism on each $V_{n,2}$, it is well known that the Grassmannian $V_{n,2}/\mathrm{O}(2)$ of unoriented 2-planes in $\mathbb{R}^n$ is orientable if and only if n is even (see for instance [35, Example 47 on page 162]). We show that a similar—but *shifted*—result holds when $\mathrm{O}(2)$ is replaced by D_8.

Notation 3.1. For a subgroup H of $\mathrm{O}(2)$, we will use the shorthand $V_{n,H}$ to denote the quotient $V_{n,2}/H$. For instance $V_{m+1,G} = E_{m,G}$, the space in Notation 1.7.

Proposition 3.2. *For $n > 2$, V_{n,D_8} is orientable if and only if n is odd. Consequently, for $m > 1$, the top dimensional cohomology group of $B(\mathrm{P}^m, 2)$ is*

$$H^{2m-1}(B(\mathrm{P}^m,2)) = \begin{cases} \mathbb{Z}, & \textit{for even } m; \\ \mathbb{Z}_2, & \textit{for odd } m. \end{cases}$$

Remark 3.3. Proposition 3.2 holds (with the same proof) if D_8 is replaced by its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $B(\mathrm{P}^m, 2)$ is replaced by $F(\mathrm{P}^m, 2)$. It is interesting to compare both versions of Proposition 3.2 with the fact that, for $m > 1$, $B(\mathrm{P}^m, 2)$ is non-orientable, while $F(\mathrm{P}^m, 2)$ is orientable only for odd m ([26, Lemma 2.6]).

Example 3.4. The cases with $n = 2$ and $m = 1$ in Proposition 3.2 are special (compare to [26, Proposition 2.5]): Since the quotient of $V_{2,2} = S^1 \cup S^1$ by the action of $D_8 \cap \mathrm{SO}(2)$ is diffeomorphic to the disjoint union of two copies of $S^1/\mathbb{Z}_4$, we see that $V_{2,D_8} \cong S^1$.

If we take the same orientation for both circles in $V_{2,2} = S^1 \cup S^1$, it is clear that the automorphism $H^1(V_{2,2}) \to H^1(V_{2,2})$ induced by an element $r \in D_8$ is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if $r \in \mathrm{SO}(2)$, but by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ if $r \notin \mathrm{SO}(2)$. For larger values of n, the method of proof of Proposition 3.2 allows us to describe the action of D_8 on the integral cohomology ring of $V_{n,2}$. The answer is given in terms of the generators $\rho, \rho_1, \rho_2 \in D_8$ introduced in Definition 1.6.

Theorem 3.5. *The three automorphisms $\rho^*, \rho_1^*, \rho_2^* \colon H^q(V_{n,2}) \to H^q(V_{n,2})$ agree. For $n > 2$, this common morphism is the identity except when n is even and $q \in \{n-2, 2n-3\}$, in which case the common morphism is multiplication by -1.*

Theorem 3.5 should be read keeping in mind the well-known cohomology ring $H^*(V_{n,2})$. We recall its simple description after proving Proposition 3.2. For the time being it suffices to recall, for the purposes of Proposition 3.6 below, that $H^{n-1}(V_{n,2}) = \mathbb{Z}_2$ for odd n, $n \geq 3$.

We use our approach to Theorem 3.5 in order to describe the integral cohomology ring of the oriented Grassmannian $V_{n,\mathrm{SO}(2)}$ for odd n, $n \geq 3$. Although the result might be well known ($V_{n,\mathrm{SO}(2)}$ is a complex quadric of complex dimension $n-2$), we include the details (an easy step from the constructions in this section) since we have not been able to find an explicit reference in the literature.

Proposition 3.6. *Assume n is odd, $n = 2a+1$ with $a \geq 1$. Let $\widetilde{z} \in H^2(V_{n,\mathrm{SO}(2)})$ stand for the Euler class of the smooth principal S^1-bundle*

$$S^1 \to V_{n,2} \to V_{n,\mathrm{SO}(2)} \tag{7}$$

There is a class $\widetilde{x} \in H^{n-1}(V_{n,\mathrm{SO}(2)})$ mapping under the projection in (7) to the nontrivial element in $H^{n-1}(V_{n,2})$. Furthermore, as a ring, $H^*(V_{n,\mathrm{SO}(2)}) = \mathbb{Z}[\widetilde{x}, \widetilde{z}]/I_n$ where I_n is the ideal generated by

$$\widetilde{x}^{\,2}, \quad \widetilde{x}\,\widetilde{z}^{\,a}, \quad \textit{and} \quad \widetilde{z}^{\,a} - 2\widetilde{x}. \tag{8}$$

It should be noted that the second generator of I_n is superfluous. We include it in the description since it will become clear, from the proof of Proposition 3.6, that the first two terms in (8) correspond to the two families of differentials in the SSS of the fibration classifying (7), while the last term corresponds to the family of nontrivial extensions in the resulting E_∞-term.

Remark 3.7. It is illuminating to compare Proposition 3.6 with H. F. Lai's computation of the cohomology ring $H^*(V_{n,\mathrm{SO}(2)})$ for even n, $n \geq 4$. According to [28, Theorem 2], $H^*(V_{2a,\mathrm{SO}(2)}) = \mathbb{Z}[\kappa, \widetilde{z}]/I_{2a}$ where I_{2a} is the ideal generated by

$$\kappa^2 - \varepsilon\kappa\widetilde{z}^{\,a-1} \quad \text{and} \quad \widetilde{z}^{\,a} - 2\kappa\widetilde{z}. \tag{9}$$

Here $\varepsilon = 0$ for a even, and $\varepsilon = 1$ for a odd, while the generator $\kappa \in H^{2a-2}(V_{2a,\mathrm{SO}(2)})$ is the Poincaré dual of the homology class represented by the canonical (realification) embedding $\mathbb{C}\mathrm{P}^{a-1} \hookrightarrow V_{2a,\mathrm{SO}(2)}$ (Lai also proves that $(-1)^{a-1}\kappa\widetilde{z}^{\,a-1}$ is the top dimensional cohomology class in $V_{2a,\mathrm{SO}(2)}$ corresponding to the canonical orientation of this manifold). The first fact to observe in Lai's description of $H^*(V_{2a,\mathrm{SO}(2)})$ is that the two dimensionally forced relations $\kappa\widetilde{z}^{\,a} = 0$ and $\widetilde{z}^{\,2a-1} = 0$ can be algebraically deduced from the relations implied by (9). A similar situation holds for $H^*(V_{2a+1,\mathrm{SO}(2)})$, where the first two relations in (8), as well as the corresponding algebraically implied relation $\widetilde{z}^{\,2a} = 0$, are forced by dimensional considerations. But it is more interesting to compare Lai's result with Proposition 3.6

through the canonical inclusions $\iota_n\colon V_{n,\mathrm{SO}(2)} \hookrightarrow V_{n+1,\mathrm{SO}(2)}$ $(n \geq 3)$. In fact, the relations given by the last element both in (8) and (9) readily give

$$\iota_{2a}^*(\widetilde{x}) = \kappa\widetilde{z} \quad \text{and} \quad \iota_{2a+1}^*(\kappa) = \widetilde{x} \tag{10}$$

for $a \geq 2$. Note that the second equality in (10) can be proved, for all $a \geq 1$, with the following alternative argument: From [28, Theorem 2], $2\kappa - \widetilde{z}^a \in V_{2a+2,\mathrm{SO}(2)}$ is the Euler class of the canonical *normal* bundle of $V_{2a+2,\mathrm{SO}(2)}$ and, therefore, maps trivially under ι_{2a+1}^*. The second equality in (10) then follows from the relation implied by the last element in (8). Needless to say, the usual cohomology ring $H^*(B\mathrm{SO}(2))$ is recovered as the inverse limit of the maps ι_n^* (of course $B\mathrm{SO}(2) \simeq \mathbb{C}\mathrm{P}^\infty$).

Proof of Proposition 3.2 *from Theorem* 3.5*.* Since the action of every element in $D_8 \cap \mathrm{SO}(2)$ preserves orientation in $V_{n,2}$, and since two elements in $D_8 - \mathrm{SO}(2)$ must "differ" by an orientation-preserving element in D_8, the first assertion in Proposition 3.2 will follow once we argue that (say) ρ is orientation-preserving precisely when n is odd. But such a fact is given by Theorem 3.5 in view of the UCT. The second assertion in Proposition 3.2 then follows from Lemma 1.8, [24, Corollary 3.28], and the UCT (recall $\dim(V_{n,2}) = 2n-3$). □

We now start working toward the proof of Theorem 3.5, recalling in particular the cohomology ring $H^*(V_{n,2})$. Let $n > 2$ and think of $V_{n,2}$ as the sphere bundle of the tangent bundle of S^{n-1}. The (integral cohomology) SSS for the fibration $S^{n-2} \xrightarrow{\iota} V_{n,2} \xrightarrow{\pi} S^{n-1}$ (where $\pi(v_1, v_2) = v_1$ and $\iota(w) = (e_1, (0, w))$ with $e_1 = (1, 0, \ldots, 0)$) starts as

$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & (p,q) \in \{(0,0), (n-1,0), (0,n-2), (n-1,n-2)\}; \\ 0, & \text{otherwise;} \end{cases} \tag{11}$$

and the only possibly nonzero differential is multiplication by the Euler characteristic of S^{n-1} (see for instance [32, pages 153–154]). At any rate, the only possibilities for a nonzero cohomology group $H^q(V_{n,2})$ are $\mathbb{Z}_2$ or $\mathbb{Z}$. In the former case, any automorphism must be the identity. So the real task is to determine the action of the three elements in Theorem 3.5 on a cohomology group $H^q(V_{n,2}) = \mathbb{Z}$.

Proof of Theorem 3.5*.* The fact that $\rho^* = \rho_1^* = \rho_2^*$ follows by observing that the product of any two of the elements ρ, ρ_1, and ρ_2 lies in the path connected group $\mathrm{SO}(2)$, and therefore determines an automorphism $V_{n,2} \to V_{n,2}$ which is homotopic to the identity.

The analysis of the second assertion of Theorem 3.5 depends on the parity of n.

Case with n even, $n > 2$. The SSS (11) collapses, giving that $H^*(V_{n,2})$ is an exterior algebra (over $\mathbb{Z}$) on a pair of generators x_{n-2} and x_{n-1} (indices denote dimensions). The spectral sequence also gives that x_{n-2} maps under ι^* to the generator in S^{n-2}, whereas

x_{n-1} is the image under π^* of the generator in S^{n-1}. Now, the (obviously) commutative diagram

$$\begin{tikzcd}
S^{n-2} \arrow[rr, "\text{antipodal map}"] \arrow[d, "\iota"'] & & S^{n-2} \arrow[d, "\iota"] \\
V_{n,2} \arrow[rr, "\rho_2"] \arrow[dr, "\pi"'] & & V_{n,2} \arrow[dl, "\pi"] \\
& S^{n-1} &
\end{tikzcd}$$

implies that ρ_2^* (and therefore ρ_1^* and ρ^*) is the identity on $H^{n-1}(V_{n,2})$, and that ρ_2^* (and therefore ρ_1^* and ρ^*) act by multiplication by -1 on $H^{n-2}(V_{n,2})$. The multiplicative structure then implies that the last assertion holds also on $H^{2n-3}(V_{n,2})$.

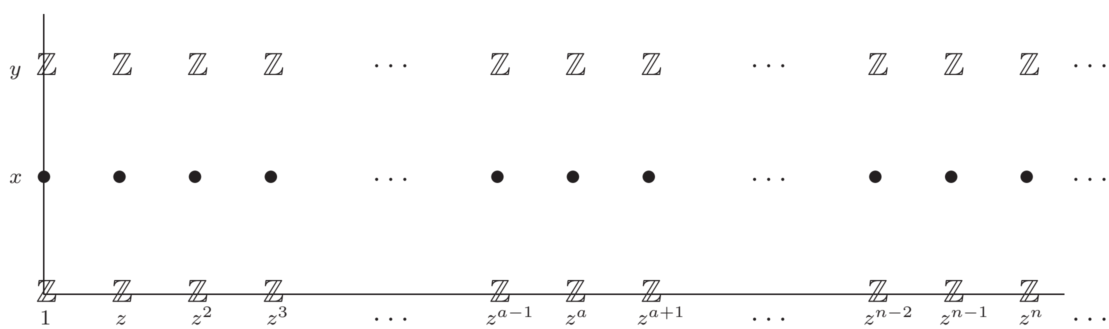
Case with n odd, $n > 2$. The description in (11) of the start of the SSS implies that the only nonzero cohomology groups of $V_{n,2}$ are $H^{n-1}(V_{n,2}) = \mathbb{Z}_2$ and $H^i(V_{n,2}) = \mathbb{Z}$ for $i = 0, 2n-3$. Thus, we only need to make sure that

$$\rho^*\colon H^{2n-3}(V_{n,2}) \to H^{2n-3}(V_{n,2}) \text{ is the identity morphism.} \tag{12}$$

Choose generators $x \in H^{n-1}(V_{n,2})$, $y \in H^{2n-3}(V_{n,2})$, and $z \in H^2(\mathbb{C}\mathrm{P}^\infty)$, and let $V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty$ classify the circle fibration (7). Thus, the E_2-term of the SSS for the fibration

$$V_{n,2} \to V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty \tag{13}$$

takes the simple form



where $n = 2a+1$, and a bullet represents a copy of $\mathbb{Z}_2$. The proof of Proposition 3.6 below gives two rounds of differentials, both originating on the top horizontal line; the element $2y$ is a cycle in the first round of differentials, but determines the second round of differentials by

$$d_{2n-2}(2y) = z^{n-1}. \tag{14}$$

The key ingredient comes from the observation that ρ and the involution $\tau\colon V_{n,\mathrm{SO}(2)} \to V_{n,\mathrm{SO}(2)}$ that reverses orientation of an oriented 2-plane fit into the pull-back diagram

$$\begin{tikzcd}
V_{n,2} \arrow[r, "\rho"] \arrow[d] & V_{n,2} \arrow[d] \\
V_{n,\mathrm{SO}(2)} \arrow[r, "\tau"] \arrow[d] & V_{n,\mathrm{SO}(2)} \arrow[d] \\
\mathbb{C}\mathrm{P}^\infty \arrow[r, "c"] & \mathbb{C}\mathrm{P}^\infty
\end{tikzcd} \tag{15}$$

where c stands for conjugation. [Indeed, thinking of $V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty$ as an inclusion, τ is the restriction of c, and ρ becomes the equivalence induced on (selected) fibers.] Of course $c^*(z) = -z$ in $H^2(\mathbb{C}\mathrm{P}^\infty)$, so that

$$c^*(z^{n-1}) = z^{n-1} \tag{16}$$

(recall n is odd). Thus, in terms of the map of spectral sequences determined by (15), conditions (14) and (16) force the relation $\rho^*(2y) = 2y$. This gives (12). □

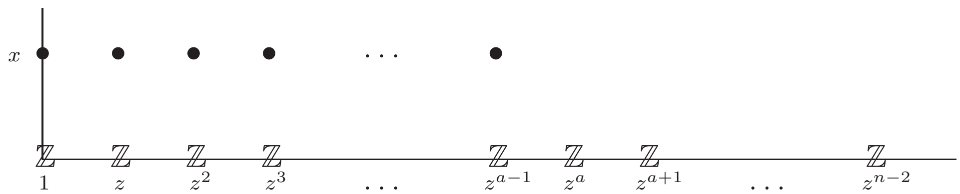
The proof of (12) we just gave (for odd n) can be simplified by working over the rationals (see Remark 3.8 in the next paragraph). We have chosen the spectral sequence analysis of (13) since it leads us to Proposition 3.6.

Remark 3.8. It is well known that whenever a finite group H acts freely on a space X, with $Y = X/H$, the rational cohomology of Y maps isomorphically onto the H-invariant elements in the rational cohomology of X (see for instance [24, Proposition 3G.1]). We apply this fact to the 8-fold covering projection $\theta\colon V_{n,2} \to V_{n,D_8}$. Since the only nontrivial groups $H^q(V_{n,2};\mathbb{Q})$ are $\mathbb{Q}$ for $q = 0, 2n-3$ (this is where we use that n is odd), we get that the rational cohomology of V_{n,D_8} is $\mathbb{Q}$ in dimension 0, vanishes in positive dimensions below $2n-3$, and is either $\mathbb{Q}$ or 0 in the top dimension $2n-3$. But V_{n,D_8} is a manifold of odd dimension, so its Euler characteristic is zero; this forces the top rational cohomology to be $\mathbb{Q}$. Thus, every element in D_8 acts as the identity on the top rational (and therefore integral) cohomology group of $V_{n,2}$. This gives in particular (12), the real content of Theorem 3.5 for an odd n.

As in the notation introduced right after (12), let $z \in H^2(\mathbb{C}\mathrm{P}^\infty)$ be a generator so that the element $\widetilde{z} \in H^2(V_{n,\mathrm{SO}(2)})$ in Proposition 3.6 is the image of z under the projection map in (13).

Proof of Proposition 3.6. The E_2-term of the SSS for (13) has been indicated in the proof of Theorem 3.5. In that picture, the horizontal x-line consists of permanent cycles; indeed, there is no nontrivial target in a $\mathbb{Z}$ group for a differential originating at a $\mathbb{Z}_2$ group. Since $\dim(V_{n,\mathrm{SO}(2)}) = 2n-4$, the term xz^a must be killed by a differential, and the only way this can happen is by means of $d_{n-1}(y) = xz^a$. By multiplicativity, this settles a whole

family of differentials killing off the elements xz^i with $i \geq a$. Note that this still leaves groups $2 \cdot \mathbb{Z}$ in the y-line (rather, the $2y$-line). Just as before, dimensionality forces the differential (14), and multiplicativity determines a corresponding family of differentials. What remains in the SSS after these two rounds of differentials—depicted below—consists of permanent cycles, so the spectral sequence collapses from this point on.



Finally, we note that all possible extensions are nontrivial. Indeed, orientability of $V_{n,\mathrm{SO}(2)}$ gives $H^{2n-4}(V_{n,\mathrm{SO}(2)}) = \mathbb{Z}$, which implies a nontrivial extension involving xz^{a-1} and z^{n-2}. Since multiplication by z is monic in total dimensions less that $2n-4$ of the E_∞-term, the 5-Lemma (applied recursively) shows that the same assertion is true in $H^*(V_{n,\mathrm{SO}(2)})$. This forces the corresponding nontrivial extensions in degrees lower than $2n-4$: an element of order 2 in low dimensions would produce, after multiplication by z, a corresponding element of order 2 in the top dimension. The proposition follows. $\square$

Lai's description of the ring $H^*(V_{2a,\mathrm{SO}(2)})$ given in Remark 3.7 can be used to understand the full patter of differentials and extensions in the SSS of (13) for $n = 2a$. Due to space limitations, details are not given here—but they are discussed in Remark 3.10 of the preliminary version [19] of this paper.

We close this section with an argument that explains, in a geometric way, the switch in parity of n when comparing the orientability properties of $V_{n,\mathrm{O}(2)}$ to those of V_{n,D_8}. Let π stand for the projection map in the smooth fiber bundle (7). The tangent bundle $T_{n,2}$ to $V_{n.2}$ decomposes as the Whitney sum

$$T_{n,2} \cong \pi^*(T_{n,\mathrm{SO}(2)}) \oplus \lambda$$

where $T_{n,\mathrm{SO}(2)}$ is the tangent bundle to $V_{n,\mathrm{SO}(2)}$, and λ is the 1-dimensional bundle of tangents to the fibers—a trivial bundle since we have the nowhere vanishing vector field obtained by differentiating the free action of S^1 on $V_{n.2}$. Note that $\rho\colon V_{n,2} \to V_{n,2}$ reverses orientation on all fibers and so reverses a given orientation of λ. Hence, ρ *preserves* a chosen orientation of $T_{n,2}$ precisely when the involution τ in (15) *reverses* a chosen orientation of $T_{n,\mathrm{SO}(2)}$. But, as explained in the proof of Proposition 3.2, V_{n,D_8} is orientable precisely when ρ is orientation-preserving. Likewise, $V_{n,\mathrm{O}(2)}$ is orientable precisely when τ is orientation-preserving.

4 Torsion linking form and Theorems 1.1 and 1.2

In this short section we outline an argument, based on the classical torsion linking form, that allows us to compute the cohomology groups described by Theorems 1.1 and 1.2 in all but three critical dimensions. The totality of dimensions (together with the proofs of Proposition 1.11 and Theorems 1.9 and 1.10) is considered in the next three sections—the first two of which represent, together with the final Section 8, the bulk of spectral sequence computations in this paper.

For a space X let $TH_i(X;A)$ (respectively, $TH^i(X;A)$) denote the torsion subgroup of the i^{th} homology (respectively, cohomology) group of X with (possibly twisted) coefficients A. As usual, omission of A from the notation indicates that a simple system of $\mathbb{Z}$-coefficients is used. We are interested in the twisted coefficients $\widetilde{\mathbb{Z}}$ arising from the orientation character of a closed m-manifold $X = M$ for, in such a case, there are non-singular pairings

$$TH^i(M) \times TH^j(M;\widetilde{\mathbb{Z}}) \to \mathbb{Q}/\mathbb{Z} \tag{17}$$

(for $i + j = m + 1$), the so-called torsion linking forms, constructed from the UCT and Poincaré duality. Although (17) seems to be best known for an orientable M (see for instance [41, pages 16–17 and 58–59]), the construction works just as well in a non-orientable setting. We briefly recall the details (in cohomological terms) for completeness.

Start by observing that for a finitely generated abelian group $H = F \oplus T$ with F free abelian and T a finite group, the group $\mathrm{Ext}^1(H,\mathbb{Z}) \cong \mathrm{Ext}^1(T,\mathbb{Z})$ is canonically isomorphic to $\mathrm{Hom}(T,\mathbb{Q}/\mathbb{Z})$, the Pontryagin dual of T (verify this by using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, and noting that $\mathbb{Q}$ is injective while $\mathrm{Hom}(T,\mathbb{Q}) = 0$). In particular, the canonical isomorphism $TH^i(M) \cong \mathrm{Ext}^1(TH_{i-1}(M),\mathbb{Z})$ coming from the UCT yields a non-singular pairing $TH^i(M) \times TH_{i-1}(M) \to \mathbb{Q}/\mathbb{Z}$. The form in (17) then follows by using Poincaré duality (in its not necessarily orientable version, see [24, Theorem 3H.6] or [36, Theorem 4.51]). As explained by Barden in [2, Section 0.7] (in the orientable case), the resulting pairing can be interpreted geometrically as the classical torsion linking number ([27, 39, 43]).

Recall the group G and orbit space E_m in Notation 1.7. We next indicate how the isomorphisms

$$TH^i(M) \cong TH^j(M;\widetilde{\mathbb{Z}}), \quad i + j = 2m, \tag{18}$$

coming from (17) for $M = E_m$ can be used for computing most of the integral cohomology groups of $F(\mathrm{P}^m, 2)$ and $B(\mathrm{P}^m, 2)$.

Since $V_{m+1,2}$ is $(m-2)$-connected[7], the map in (4) is $(m-1)$-connected. Therefore it induces an isomorphism (respectively, monomorphism) in cohomology with any—possibly

[7] Low dimensional cases with $m \leq 3$ are given special attention in Example 5.1, Remark 5.4, and (34) in the following sections.

twisted, in view of [44, Theorem 6.4.3*]—coefficients in dimensions $i \leq m-2$ (respectively, $i = m-1$). Together with Corollary 2.4 and Lemmas 1.8 and 2.8, this leads to the explicit description of the groups in Theorems 1.1 and 1.2 in dimensions at most $m-2$. The corresponding groups in dimensions at least $m+2$ can then be obtained from the isomorphisms (18) and the full description in Section 2 of the twisted and untwisted cohomology groups of BG. Note that the last step requires knowing that, when E_m is non-orientable (as determined in Proposition 3.2 and Remark 3.3), the twisted coefficients $\widetilde{\mathbb{Z}}$ agree with those $\mathbb{Z}_\alpha$ used in Theorem 2.3. But such a requirement is a direct consequence of Theorem 3.5. Since the torsion-free subgroups of $H^*(E_m)$ are easily identifiable from a quick glance at the E_2-term of the CLSS for the G-action on $V_{m+1,2}$, only the torsion subgroups in Theorems 1.1 and 1.2 in dimensions

$$m-1,\ m,\ \text{and}\ \ m+1 \tag{19}$$

are lacking description in this argument.

A deeper analysis of the CLSS of the G-action on $V_{m+1,2}$ (worked out in Sections 5 and 6 for $G = D_8$, and discussed briefly in Section 7 for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$) will give us (among other things) a detailed description of the three missing cases in (19) *except* for the $(m+1)$-dimensional group when $G = D_8$ and $m \equiv 3 \bmod 4$. Note that this apparently singular case cannot be handled directly with the torsion linking form argument in the previous paragraph because the connectivity of $V_{m+1,2}$ only gives the injectivity, but not the surjectivity, of the first map in the composite

$$H^{m-1}(BD_8;\mathbb{Z}_\alpha) \xrightarrow{p^*} H^{m-1}(B(\mathrm{P}^m,2);\mathbb{Z}_\alpha) \cong H^{m+1}(B(\mathrm{P}^m,2)). \tag{20}$$

To overcome the problem, in Section 6 we perform a direct calculation in the first two pages of the Bockstein spectral sequence (BSS) of $B(\mathrm{P}^{4a+3},2)$ to prove that (20) is indeed an isomorphism for $m \equiv 3 \bmod 4$—therefore completing the proof of Theorems 1.1 and 1.2.

$*=$	2	3	4	5	6	7	8	9	10	11	12	13	14
$H^*(E_{2,D_8})$	$\langle 2\rangle$												
$H^*(E_{4,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$								
$H^*(E_{6,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 2\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$				
$H^*(E_{8,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 2\rangle$	$\langle 4\rangle$	$\langle 3\rangle$	$\{4\}$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$

Table 1: $H^*(E_{m,D_8}) \cong H^*(B(\mathrm{P}^m,2))$ for $m = 2,4,6,$ and 8

The isomorphisms in (18) yield a (twisted, in the non-orientable case) symmetry for the torsion groups of $H^*(E_m)$. This is illustrated (for $G = D_8$ and in the orientable case) in Table 1 following the conventions set in the very first paragraph of the paper.

5 Case of $B(\mathrm{P}^m, 2)$ for $m \not\equiv 3 \bmod 4$

This section and the next one contain a careful study of the CLSS of the D_8-action on $V_{m+1,2}$ described in Definition 1.6; the corresponding (much simpler) analysis for the restricted $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action is outlined in Section 7. The CLSS approach will yield, in addition, direct proofs of Proposition 1.11 and Theorems 1.9 and 1.10. The reader is assumed to be familiar with the properties of the CLSS of a regular covering space, complete details of which first appeared in [6].

We start with the less involved situation of an even m and, as a warm-up, we consider first the case $m = 2$.

Example 5.1. Lemmas 1.8 and 2.1, Corollary 2.4, and Theorem 3.5 imply that, in total dimensions at most $\dim(V_{3,D_8}) = 3$, the (integral cohomology) CLSS for the D_8-action on $V_{3,2}$ starts as

$$
\begin{array}{c|cccc}
3 & \mathbb{Z} & & & \\
2 & \langle 1\rangle & \langle 2\rangle & & \\
1 & & & & \\
 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle \\
\hline
 & & 1 & 2 & 3
\end{array}
$$

The only possible nontrivial differential in this range is $d_3^{0,2}\colon E_2^{0,2} \to E_2^{3,0}$, which must be an isomorphism in view of the second assertion in Proposition 3.2. This yields the P^2-case in Proposition 1.11 and Theorems 1.2 and 1.9 (with $G = D_8$ in the latter one). As indicated in Table 1, the symmetry isomorphisms are invisible in the current situation. It is worth noticing that the d_3-differential originating at node $(1, 2)$ must be injective. This observation will be the basis in our argument for the general situation, where 2-rank considerations will be the catalyst. Here and in what follows, by the 2-*rank* (or simply *rank*) of a finite abelian 2-group H we mean the rank ($\mathbb{F}_2$-dimension) of $H \otimes \mathbb{F}_2$.

Proof of Theorem 1.9 *for* $G = D_8$*, and of Proposition* 1.11*, both with* m *even,* $m \geq 4$. The assertion in Theorem 1.9 for

- $i \geq 2m$ follows from Lemma 1.8 and the fact that $\dim(V_{m+1,2}) = 2m - 1$, and for
- $i = 2m - 1$ follows from the fact that $H^{2m-1}(BD_8)$ is a torsion group (Corollary 2.4) while $H^{2m-1}(B(\mathrm{P}^m, 2)) = \mathbb{Z}$ (Proposition 3.2).

We work with the (integral cohomology) CLSS for the D_8-action on $V_{m+1,2}$ in order to prove Propostion 1.11 and the assertions in Theorem 1.9 for $i < 2m - 1$.

In view of Theorem 3.5, the spectral sequence has a simple system of coefficients and, from the description of $H^*(V_{m+1,2})$ in the proof of Theorem 3.5, it is concentrated in the three horizontal lines with $q = 0, m, 2m-1$. We can focus on the lines with $q = 0, m$ in view of the range under current consideration. At the start of the CLSS there is a copy of

- $H^*(BD_8)$ (described by Corollary 2.4) at the line with $q = 0$;
- $H^*(BD_8, \mathbb{F}_2)$ (described by Lemma 2.1) at the line with $q = m$.

Note that the assertion in Theorem 1.9 for $i < m$ is an obvious consequence of the above description of the E_2-term of the CLSS. The case $i = m$ will follow once we show that the "first" potentially nontrivial differential $d_{m+1}^{0,m}\colon E_2^{0,m} \to E_2^{m+1,0}$ is injective. More generally, we show in the paragraph following (24) below that

$$\text{all differentials } d_{m+1}^{m-\ell-1,m}\colon E_2^{m-\ell-1,m} \to E_2^{2m-\ell,0} \text{ with } 0 < \ell < m \text{ are injective.} \tag{21}$$

From this, the assertion in Theorem 1.9 for $m < i < 2m-1$ follows at once.

The information we need about differentials is forced by the "size" of their domains and codomains. For instance, since $H^{2m-1}(B(\mathrm{P}^m, 2))$ is torsion-free, all of $E_2^{2m-1,0} = H^{2m-1}(BD_8) = \langle m-1 \rangle$ must be killed by differentials. But the only possibly nontrivial differential landing in $E_2^{2m-1,0}$ is the one in (21) with $\ell = 1$. The resulting surjective $d_{m+1}^{m-2,m}$ map must be an isomorphism since its domain, $E_2^{m-2,m} = H^{m-2}(BD_8; \mathbb{F}_2) = \langle m-1 \rangle$, is isomorphic to its codomain.

The extra input we need in order to deal with the rest of the differentials in (21) comes from the short exact sequences

$$0 \to \mathrm{Coker}(2_i) \to H^i(B(\mathrm{P}^m, 2); \mathbb{F}_2) \to \mathrm{Ker}(2_{i+1}) \to 0 \tag{22}$$

obtained from the Bockstein long exact sequence

$$\cdots \leftarrow H^i(B(\mathrm{P}^m, 2); \mathbb{F}_2) \xleftarrow{\pi_i} H^i(B(\mathrm{P}^m, 2)) \xleftarrow{2_i} H^i(B(\mathrm{P}^m, 2)) \xleftarrow{\partial_i} H^{i-1}(B(\mathrm{P}^m, 2); \mathbb{F}_2) \leftarrow \cdots.$$

From the E_2-term of the spectral sequence we easily see that ($H^1(B(\mathrm{P}^m, 2)) = 0$ and that) $H^i(B(\mathrm{P}^m, 2))$ is a finite 2-torsion group for $1 < i < 2m-1$; let r_i denote its 2-rank. Then $\mathrm{Ker}(2_i) \cong \mathrm{Coker}(2_i) \cong \langle r_i \rangle$, so that (22), Corollary 2.2, and an easy induction (grounded by the fact that $\mathrm{Ker}(2_{2m-1}) = 0$, in view of the second assertion in Proposition 3.2) yield

$$r_{2m-\ell} = \begin{cases} a+1, & \ell = 2a; \\ a, & \ell = 2a+1; \end{cases} \tag{23}$$

for $2 \le \ell \le m-1$. Under these conditions, the ℓ-th differential in (21) takes the form

$$\langle m-\ell \rangle = H^{m-\ell-1}(BD_8; \mathbb{F}_2) \to H^{2m-\ell}(BD_8) = \begin{cases} \left\{m - \frac{\ell}{2}\right\}, & \ell \equiv 0 \bmod 4; \\ \left\langle m - \frac{\ell-2}{2} \right\rangle, & \ell \equiv 2 \bmod 4; \\ \left\langle m - \frac{\ell+1}{2} \right\rangle, & \text{otherwise.} \end{cases} \tag{24}$$

But the cokernel of this map, which is a subgroup of $H^{2m-\ell}(B(\mathrm{P}^m,2))$, must have 2-rank at most $r_{2m-\ell}$. An easy counting argument (using the right exactness of the tensor product) shows that this is possible only with an injective differential (24) which, in the case of $\ell \equiv 0 \bmod 4$, yields an injective map even after tensoring[8] with $\mathbb{Z}_2$.

Note that, in total dimensions at most $2m-2$, the E_{m+2}-term of the spectral sequence is concentrated on the base line ($q=0$). Thus, for $2 \le \ell \le m-1$, $H^{2m-\ell}(B(\mathrm{P}^m,2))$ is the cokernel of the differential (24)—which yields the surjectivity asserted in Theorem 1.9 in the range $m < i < 2m-1$. Furthermore the kernel of $p^*\colon H^{2m-\ell}(BD_8) \to H^{2m-\ell}(B(\mathrm{P}^m,2))$ is the elementary abelian 2-group specified on the left hand side of (24). In fact, the observation in the second half of the final assertion in the previous paragraph proves Proposition 1.11. $\square$

As indicated in the last paragraph of the previous proof, for $2 \le \ell \le m-1$ the CLSS analysis identifies the group $H^{2m-\ell}(B(\mathrm{P}^m,2))$ as the cokernel of (24). Thus, the following algebraic calculation of these groups not only gives us an alternative approach to that using the non-singularity of the torsion linking form, but it also allows us to recover (for m even and $G = D_8$) the three missing cases in (19)—therefore completing the proof of the $\mathrm{P}^{\mathrm{even}}$-case of Theorem 1.2.

Proposition 5.2. *For* $2 \le \ell \le m-1$*, the cokernel of the differential* (24) *is isomorphic to*

$$H^{2m-\ell}(B(\mathrm{P}^m,2)) = \begin{cases} \left\{\frac{\ell}{2}\right\}, & \ell \equiv 0 \bmod 4; \\ \left\langle\frac{\ell}{2}+1\right\rangle, & \ell \equiv 2 \bmod 4; \\ \left\langle\frac{\ell-1}{2}\right\rangle, & \textit{otherwise.} \end{cases}$$

Proof. Cases with $\ell \not\equiv 0 \bmod 4$ follow from a simple count, so we only offer an argument for $\ell \equiv 0 \bmod 4$. Consider the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \langle m-\ell\rangle \arrow[r] & \left\{m-\frac{\ell}{2}\right\} \arrow[r] & H^{2m-\ell}(B(\mathrm{P}^m,2)) \arrow[r] & 0 \\ 0 \arrow[r] & \langle m-\ell\rangle \arrow[r] \arrow[u, equal] & \left\langle m-\frac{\ell}{2}+1\right\rangle \arrow[r] \arrow[u, hook] & \left\langle\frac{\ell}{2}+1\right\rangle \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

where the top horizontal monomorphism is (24), and where the middle group on the bottom is included in the top one as the elements annihilated by multiplication by 2. The lower right group is $\langle\frac{\ell}{2}+1\rangle$ by a simple counting. The snake lemma shows that the right-hand-side vertical map is injective with cokernel $\mathbb{Z}_2$; the resulting extension is nontrivial in view of (23). $\square$

[8]This amounts to the fact that twice the generator of the $\mathbb{Z}_4$-summand in (24) is not in the image of (24)—compare to the proof of Proposition 5.2.

Example 5.3. For m even, [5, Theorem 1.4 (D)] identifies three explicit elements in the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$: one for each of $i = m+2$, $i = m+3$, and $i = m+4$. In particular, this produces at most four basis elements in the ideal $\mathrm{Ker}(p^*)$ in dimensions at most $m+4$. However we have just seen that, for $m+1 \leq i \leq 2m-1$, the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$ is an $\mathbb{F}_2$-vector space of dimension $i-m$. This means that through dimensions at most $m+4$ (and with $m > 4$) there are at least six more basis elements remaining to be identified in $\mathrm{Ker}(p^*)$.

We next turn to the case when m is odd (a hypothesis in force throughout the rest of the section) assuming, from Lemma 5.5 on, that $m \equiv 1 \bmod 4$.

Remark 5.4. Since the P^1-case in Proposition 1.11 and Theorems 1.2 and 1.10 is elementary (in view of Remark 1.3 and Corollary 2.4), we will implicitly assume $m \neq 1$.

The CLSS of the D_8-action on $V_{m+1,2}$ now has a few extra complications that turn the analysis of differentials into a harder task. To begin with, we find a twisted system of local coefficients (Theorem 3.5). As a $\mathbb{Z}[D_8]$-module, $H^q(V_{m+1,2})$ is:

- $\mathbb{Z}$ for $q = 0, m$;
- $\mathbb{Z}_\alpha$ for $q = m-1, 2m-1$;
- the zero module otherwise.

Thus, in total dimensions at most $2m-2$ the CLSS is concentrated on the three horizontal lines with $q = 0, m-1, m$. [This is in fact the case in total dimensions at most $2m-1$, since $H^0(BD_8; \mathbb{Z}_\alpha) = 0$; this observation is not relevant for the actual group $H^{2m-1}(B(\mathrm{P}^m, 2)) = \mathbb{Z}_2$—given in the second assertion in Proposition 3.2—, but it will be relevant for the claimed surjectivity of the map $p^*\colon H^{2m-1}(BD_8) \to H^{2m-1}(B(\mathrm{P}^m, 2))$.] In more detail, at the start of the CLSS we have a copy of $H^*(BD_8)$ at $q = 0, m$, and a copy of $H^*(BD_8; \mathbb{Z}_\alpha)$ at $q = m-1$. It is the extra horizontal line at $q = m-1$ (not present for an even m) that leads to potential d_2-differentials—from the ($q = m$)-line to the ($q = m-1$)-line. Sorting these differentials out is the main difficulty (which we have been able to overcome only for $m \equiv 1 \bmod 4$). Throughout the remainder of the section we work in terms of this spectral sequence, making free use of the description of its E_2-term coming from Corollaries 2.4 and 2.5, as well as of its $H^*(BD_8)$-module structure. Note that the latter property implies that much of the global structure of the spectral sequence is dictated by differentials on the three elements

- $x_m \in E_2^{0,m} = H^0(BD_8; H^m(V_{m+1,2})) = H^0(BD_8; \mathbb{Z}) = \mathbb{Z}$;
- $\alpha_1 \in E_2^{1,m-1} = H^1(BD_8; H^{m-1}(V_{m+1,2})) = H^1(BD_8; \mathbb{Z}_\alpha) = \mathbb{Z}_2$;
- $\alpha_2 \in E_2^{2,m-1} = H^2(BD_8; H^{m-1}(V_{m+1,2})) = H^2(BD_8; \mathbb{Z}_\alpha) = \mathbb{Z}_4$;

each of which is a generator of the indicated group (notation is inspired by that in Theorem 2.3 and in the proof of Theorem 3.5—for even n).

Lemma 5.5. *For $m \equiv 1 \bmod 4$ and $m \geq 5$, the nontrivial d_2-differentials are given by $d_2^{4i,m}(\kappa_4^i x_m) = 2\kappa_4^i \alpha_2$ for $i \geq 0$.*

Proof. The only potentially nontrivial d_2-differentials originate at the $(q = m)$-line and, in view of the module structure, all we need to show is that

$$d_2 \colon E_2^{0,m} \to E_2^{2,m-1} \text{ has } d_2(x_m) = 2\alpha_2 \tag{25}$$

(here and in what follows we omit superscripts of differentials).

Let $m = 4a + 1$. Since $H^{2m-1}(B(\mathrm{P}^m, 2)) = \langle 1 \rangle$, most of the elements in $E_2^{2m-1,0} = \langle 4a \rangle$ must be wiped out by differentials. The only differentials landing in a $E_r^{2m-1,0}$ (that originate at a nonzero group) are

$$d_m \colon E_m^{m-1,m-1} \to E_m^{2m-1,0} \quad \text{and} \quad d_{m+1} \colon E_{m+1}^{m-2,m} \to E_{m+1}^{2m-1,0}. \tag{26}$$

But $E_2^{m-1,m-1} = \langle 2a \rangle$ and $E_2^{m-2,m} = \langle 2a - 1 \rangle$, so that rank considerations imply

$$E_2^{m-2,m} = E_{m+1}^{m-2,m}, \tag{27}$$

with the two differentials in (26) injective. In particular we get that

$$H^{2m-1}(B(\mathrm{P}^m, 2)) = \langle 1 \rangle \text{ comes from } E_\infty^{2m-1,0} = \langle 1 \rangle. \tag{28}$$

Furthermore, (27) and the $H^*(BD_8)$-module structure in the spectral sequence imply that the differential in (25) cannot be surjective.

It remains to show that the differential in (25) is nonzero. We shall obtain a contradiction by assuming that $d_2(x_m) = 0$, so that every element in the $(q = m)$-line is a d_2-cycle. Since $H^{2m}(B(\mathrm{P}^m, 2)) = 0$, all of $E_2^{2m,0} = \langle 4a + 2 \rangle$ must be wiped out by differentials, and under the current hypothesis the only possible such differentials would be $d_m \colon E_m^{m,m-1} = E_2^{m,m-1} = \langle 2a + 1 \rangle \to E_m^{2m,0} = E_2^{2m,0}$ and $d_{m+1} \colon E_{m+1}^{m-1,m} = E_2^{m-1,m} = \langle 2a \rangle \oplus \mathbb{Z}_4 \to E_{m+1}^{2m,0}$—indeed, $E_2^{0,2m-1} = H^0(BD_8; \mathbb{Z}_\alpha) = 0$. Thus, the former differential would have to be injective while the latter one would have to be surjective with a $\mathbb{Z}_2$ kernel. But there are no further differentials that could kill the resulting $E_{m+2}^{m-1,m} = \langle 1 \rangle$, in contradiction to (28). □

Remark 5.6. In the preceding proof we made crucial use of the $H^*(BD_8)$-module structure in the spectral sequence in order to handle d_2-differentials. We show next that, just as in the proof of Theorem 1.9 for $G = D_8$, many of the properties of all higher differentials in the case $m \equiv 1 \bmod 4$ follow from the "size" of the resulting E_3-term.

Proof of Theorem 1.10 *for* $G = D_8$*, and of Proposition* 1.11*, both for* $m \equiv 1 \bmod 4$. The d_2-differentials in Lemma 5.5 replace, by a $\mathbb{Z}_2$-group, every instance of a $\mathbb{Z}_4$-group in the $(q = m-1)$ and $(q = m)$-lines of the E_2-term. This describes the E_3-term, the starting stage of the CLSS in the following considerations (note that the E_3-term agrees with the E_m-term). With this information the idea of the proof is formally the same as that in the case of an even m, namely: a little input from the Bockstein long exact sequence for $B(\mathrm{P}^m, 2)$ forces the injectivity of all relevant higher differentials (we give the explicit details for the reader's benefit).

Let $m = 4a + 1$ (recall we are assuming $a \geq 1$). The crux of the matter is showing that the differentials

$$d_m\colon E_3^{m-\ell,m-1} \to E_3^{2m-\ell,0} \quad \text{with} \quad \ell = 0, 1, 2, \ldots, m \tag{29}$$

and

$$d_{m+1}\colon E_3^{m-\ell-1,m} \to E_{m+1}^{2m-\ell,0} \quad \text{with} \quad \ell = 0, 1, 2, \ldots, m-1 \tag{30}$$

are injective and never hit twice the generator of a $\mathbb{Z}_4$-group. This assertion has already been shown for $\ell = 1$ in the paragraph containing (26). Likewise, the assertion for $\ell = 0$ follows from (28) with the same counting argument as the one used in the final paragraph of the proof of Lemma 5.5. Furthermore the case $\ell = m$ in (29) is obvious since $E_3^{0,m-1} = H^0(BD_8; \mathbb{Z}_\alpha) = 0$. However, since $E_3^{0,m} = H^0(BD_8) = \mathbb{Z}$ and $E_3^{m+1,0} = H^{m+1}(BD_8) = \langle 2a+2 \rangle$, the injectivity assertion needs to be suitably interpreted for $\ell = m-1$ in (30); indeed, we will prove that

$$d_{m+1}\colon E_3^{0,m} \to E_{m+1}^{m+1,0} \quad \text{yields an injective map } \textit{after} \text{ tensoring with } \mathbb{Z}_2. \tag{31}$$

From the E_3-term of the spectral sequence we easily see that $H^m(B(\mathrm{P}^m, 2))$ is the direct sum of a copy of $\mathbb{Z}$ and a finite 2-torsion group, while $H^i(B(\mathrm{P}^m, 2))$ is a finite 2-torsion group for $i \neq 0, m$. We consider the analogue of (22), the short exact sequences

$$0 \to \mathrm{Coker}(2_i) \to H^i(B(\mathrm{P}^m, 2); \mathbb{F}_2) \to \mathrm{Ker}(2_{i+1}) \to 0, \tag{32}$$

working here and below in the range $m+1 \leq i \leq 2m-2$. Let r_i denote the 2-rank of (the torsion subgroup of) $H^i(B(\mathrm{P}^m, 2))$, so that $\mathrm{Ker}(2_i) \cong \mathrm{Coker}(2_i) \cong \langle r_i \rangle$. Then Corollary 2.2, (32), and an easy induction (grounded by the fact that $\mathrm{Ker}(2_{2m-1}) = \langle 1 \rangle$, which in turn comes from the second assertion in Proposition 3.2) yield that

$$r_{2m-\ell} \text{ is the integral part of } \tfrac{\ell+1}{2} \text{ for } 2 \leq \ell \leq m-1. \tag{33}$$

Now, in the range of (33), Lemma 5.5 and Corollaries 2.4 and 2.5 give

$$
\begin{aligned}
E_3^{m-\ell,m-1} &= \begin{cases} \left\langle 2a+1-\frac{\ell}{2}\right\rangle, & \ell \text{ even;} \\ \left\langle 2a-\frac{\ell-1}{2}\right\rangle, & \ell \text{ odd;} \end{cases} \\
E_3^{m-\ell-1,m} &= \begin{cases} \mathbb{Z}, & \ell = m-1; \\ \left\langle 2a+1-\frac{\ell}{2}\right\rangle, & \ell \text{ even, } \ell < m-1; \\ \left\langle 2a-\frac{\ell+1}{2}\right\rangle, & \ell \text{ odd;} \end{cases} \\
E_3^{2m-\ell,0} &= \begin{cases} \left\langle 4a+2-\frac{\ell}{2}\right\rangle, & \ell \equiv 0 \bmod 4; \\ \left\{ 4a+1-\frac{\ell}{2}\right\} & \ell \equiv 2 \bmod 4; \\ \left\langle 4a-\frac{\ell-1}{2}\right\rangle, & \text{otherwise;} \end{cases}
\end{aligned}
$$

and since $E_{m+2}^{2m-\ell,0}$ has 2-rank at most $r_{2m-\ell}$ (indeed, $E_{m+2}^{2m-\ell,0} = E_{\infty}^{2m-\ell,0}$ which is a subgroup of $H^{2m-\ell}(B(\mathrm{P}^m,2))$), an easy counting argument (using, as in the case of an even m, the right exactness of the tensor product) gives that the differentials in (29) and (30) must yield an injective map after tensoring with $\mathbb{Z}_2$. In particular they

(a) must be injective on the nose, except for the case discussed in (31);

(b) cannot hit twice the generator of a $\mathbb{Z}_4$-summand.

The already observed equalities $E_2^{0,2m-1} = H^0(BD_8;\mathbb{Z}_\alpha) = 0$ together with (a) above imply that, in total dimensions t with $t \leq 2m-1$ and $t \neq m$, the E_{m+2}-term of the spectral sequence is concentrated on the base line ($q = 0$), while at higher lines ($q > 0$) the spectral sequence only has a $\mathbb{Z}$-group—at node $(0, m)$. This situation yields Theorem 1.10, while (b) above yields Proposition 1.11. □

A direct calculation (left to the reader) using the proved behavior of the differentials in (29) and (30)—and using (twice) the analogue of Proposition 5.2 when $\ell \equiv 2 \bmod 4$—gives

$$
H^{2m-\ell}(B(\mathrm{P}^m,2)) = \begin{cases} \left\langle \frac{\ell}{2}\right\rangle, & \ell \equiv 0 \bmod 4; \\ \left\{ \frac{\ell}{2}-1\right\}, & \ell \equiv 2 \bmod 4; \\ \left\langle \frac{\ell+1}{2}\right\rangle, & \text{otherwise;} \end{cases}
$$

for $2 \leq \ell \leq m-1$. Thus, as the reader can easily check using Corollaries 2.4 and 2.5, instead of the symmetry isomorphisms exemplified in Table 1, the cohomology groups of $B(\mathrm{P}^m,2)$ are now formed (as predicted by the isomorphisms (18) of the previous section) by a combination of $H^*(BD_8)$ and $H^*(BD_8;\mathbb{Z}_\alpha)$—in the lower and upper halves, respectively. Once again, the CLSS analysis not only offers an alternative to the (torsion linking form) arguments in the previous section, but it allows us to recover, under the present hypotheses, the torsion subgroup in the three missing dimensions in (19).

Example 5.7. For $m \equiv 1 \bmod 4$, [5, Theorem 1.4 (D)] identifies two explicit elements in the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$: one for each of $i = m+1$ and $i = m+3$. In particular, this produces at most three basis elements in the ideal $\mathrm{Ker}(p^*)$ in dimensions at most $m+3$. However it follows from the previous spectral sequence analysis that, for $m+1 \leq i \leq 2m-1$, the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$ is an $\mathbb{F}_2$-vector space of dimension $i - m + (-1)^i$. This means that through dimensions at most $m+3$ (and with $m \geq 5$) there are at least four more basis elements remaining to be identified in $\mathrm{Ker}(p^*)$.

6 Case of $B(\mathrm{P}^{4a+3}, 2)$

We now discuss some aspects of the spectral sequence of the previous section in the unresolved case $m \equiv 3 \bmod 4$. Although we are unable to describe the pattern of differentials for such m, we show that enough information can be collected to not only resolve the three missing cases in (19), but also to conclude the proof of Theorem 1.10 for $G = D_8$. Unless explicitly stated otherwise, the hypothesis $m \equiv 3 \bmod 4$ will be in force throughout the section.

Remark 6.1. The main problem that has prevented us from fully understanding the spectral sequence of this section comes from the apparent fact that the algebraic input coming from the $H^*(BD_8)$-module structure in the CLSS—the crucial property used in the proof of Lemma 5.5—does not give us enough information in order to determine the pattern of d_2-differentials. New geometric insights seem to be needed instead. Although it might be tempting to conjecture the validity of Lemma 5.5 for $m \equiv 3 \bmod 4$, we have not found concrete evidence supporting such a possibility. In fact, a careful analysis of the possible behaviors of the spectral sequence for $m = 3$ (performed in Example 6.4 of the preliminary version [19] of this paper) does not give even a more aesthetically pleasant reason for leaning toward the possibility of having a valid Lemma 5.5 in the current congruence. A second problem arose in [19] when we noted that, even if the pattern of d_2-differentials were known for $m \equiv 3 \bmod 4$, there would seem to be a slight indeterminacy either in a few higher differentials (if Lemma 5.5 holds for $m \equiv 3 \bmod 4$), or in a few possible extensions among the $E_\infty^{p,q}$ groups (if Lemma 5.5 actually fails for $m \equiv 3 \bmod 4$). Even though we cannot resolve the current d_2-related ambiguity, in [19, Example 6.4] we note that, at least for $m = 3$, it is possible to overcome the above mentioned problems about higher differentials or possible extensions by making use of the explicit description of $H^4(B(\mathrm{P}^3, 2))$—given later in the section (considerations previous to Remark 6.3) in regard to the claimed surjectivity of (20); see also [17], where advantage is taken of the fact that P^3 is a group.

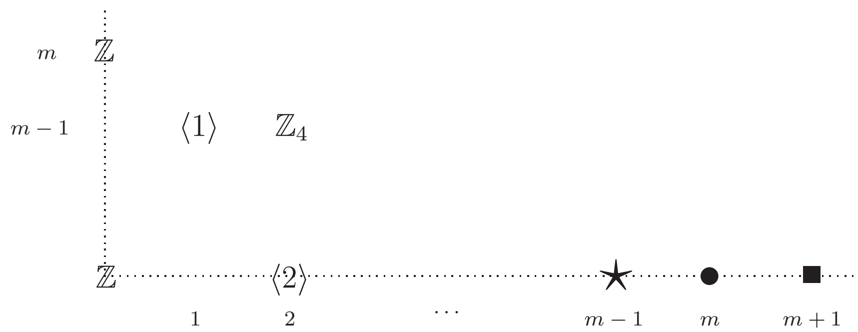
In the first result of this section, Theorem 1.10 for $G = D_8$ and $m \equiv 3 \bmod 4$, we show that, despite the previous comments, the spectral sequence approach can still be used to compute $H^*(B(\mathrm{P}^{4a+3}, 2))$ just beyond the middle dimension (i.e., just before the

first problematic d_2-differential plays a decisive role). In particular, this computes the corresponding groups in the first two of the three missing cases in (19).

Proposition 6.2. *Let $m = 4a + 3$. The map $H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$ induced by (4) is:*

1. *an isomorphism for $i < m$;*
2. *a monomorphism onto the torsion subgroup of $H^i(B(\mathrm{P}^m, 2)) = \langle 2a+1 \rangle \oplus \mathbb{Z}$ for $i = m$;*
3. *the zero map for $2m - 1 < i$.*

Proof. The argument parallels that used in the analysis of the CLSS when $m \equiv 1 \bmod 4$. Here is the chart of the current E_2-term through total dimensions at most $m + 1$:



The star at node $(m-1, 0)$ stands for $\langle 2a+2 \rangle$; the bullet at node $(m, 0)$ stands for $\langle 2a+1 \rangle$; the solid box at node $(m+1, 0)$ stands for $\{2a+2\}$. In this range there are only three possibly nonzero differentials:

- a d_2 from node $(0, m)$ to node $(2, m-1)$;
- a d_m from node $(1, m-1)$ to node $(m+1, 0)$;
- a d_{m+1} from node $(0, m)$ to node $(m+1, 0)$.

Whatever these d_2 and d_{m+1} are, there will be a resulting $E_\infty^{0,m} = \mathbb{Z}$. On the other hand, the argument about 2-ranks in (22) and in (32), leading respectively to (23) and (33), now yields that the torsion 2-group $H^{m+1}(B(\mathrm{P}^m, 2))$ has 2-rank $2a + 1$. Since $E_\infty^{m+1,0}$ is a subgroup of $H^{m+1}(B(\mathrm{P}^m, 2))$, this forces the two differentials d_m and d_{m+1} above to be nonzero, each one with cokernel of 2-rank one less than the 2-rank of its codomain. In fact, d_m must have cokernel isomorphic to $\{2a+1\}$, whereas the cokernel of d_{m+1} is either $\{2a\}$ or $\langle 2a+1 \rangle$ (Remark 6.3, and especially [19, Example 6.4], expand on these possibilities). What matters here is the forced injectivity of d_m, which implies $E_\infty^{1,m-1} = 0$ and, therefore, the second assertion of the proposition—the first assertion is obvious from the CLSS, while the third one is elementary. ∎

We now start work on the only groups in Theorem 1.2 not yet computed, namely $H^{m+1}(B(\mathrm{P}^m,2))$ for $m = 4a+3$. As indicated in the previous proof, these are torsion 2-groups of 2-rank $2a+1$. Furthermore, (20) and Corollary 2.5 show that each such group contains a copy of $\{2a\}$, a 2-group of the same 2-rank as that of $H^{m+1}(B(\mathrm{P}^m,2))$. In showing that the two groups actually agree (thus completing the proof of Theorem 1.2), a key fact comes from Fred Cohen's observation (recalled in the paragraph previous to Remark 1.3) that *there are no elements of order* 8. For instance,

$$\text{when } m = 3 \text{ the two groups must agree since both are cyclic (i.e., have 2-rank 1).} \tag{34}$$

In order to deal with the situation for positive values of a, Cohen's observation is coupled with a few computations in the first two pages of the Bockstein spectral sequence (BSS) for $B(\mathrm{P}^m,2)$: we will show that there is only one copy of $\mathbb{Z}_4$ (the one coming from the subgroup $\{2a\}$) in the decomposition of $H^{m+1}(B(\mathrm{P}^m,2))$ as a sum of cyclic 2-groups—forcing $H^{m+1}(B(\mathrm{P}^m,2)) = \{2a\}$.

Remark 6.3. Before undertaking the BSS calculations (in Proposition 6.4 below), we pause to observe that, unlike the Bockstein input in all the previous CLSS-related proofs, the use of the BSS does not seem to give quite enough information in order to understand the pattern of d_2-differentials in the current CLSS. Much of the problem lies in being able to decide the actual cokernel of the d_{m+1}-differential in the previous proof and, consequently, understand how the $\mathbb{Z}_4$-group in $H^{m+1}(B(\mathrm{P}^m,2))$ arises in the current CLSS; either entirely at the $q=0$ line (as in all cases of the previous—and the next—section), or as a nontrivial extension in the E_∞ chart (just as in the case of the spectral sequence in Section 8—see also [17, Section 9]).

Recall from [13, 21] that the mod 2 cohomology ring of $B(\mathrm{P}^m,2)$ is polynomial on three classes x, x_1, and x_2, of respective dimensions 1, 1, and 2, subject to the three relations

(I) $\quad x^2 = xx_1$;

(II) $$\sum_{0\le i\le \frac{m}{2}} \binom{m-i}{i} x_1^{m-2i} x_2^i = 0;$$

(III) $$\sum_{0\le i\le \frac{m+1}{2}} \binom{m+1-i}{i} x_1^{m+1-2i} x_2^i = 0.$$

Further, the action of Sq^1 is determined by (I) and

$$\mathrm{Sq}^1 x_2 = x_1 x_2. \tag{35}$$

[The following observations—proved in [13, 21], but not needed in this paper—might help the reader to assimilate the facts just described: The three generators x, x_1, and x_2 are in

fact the images under the map p_{m,D_8} in (4) of the corresponding classes at the beginning of Section 2. In turn, the latter generators x_1 and x_2 come from the Stiefel-Whitney classes w_1 and w_2 in $B\mathrm{O}(2)$ under the classifying map for the inclusion $D_8 \subset \mathrm{O}(2)$. In these terms, (35) corresponds to the (simplified in $B\mathrm{O}(2)$) Wu formula $\mathrm{Sq}^1(w_2) = w_1 w_2$. Finally, the two relations (II) and (III) correspond to the fact that the two dual Stiefel-Whitney classes $\overline{w}_m$ and $\overline{w}_{m+1}$ in $B\mathrm{O}(2)$ generate the kernel of the map induced by the Grassmann inclusion $G_{m+1,2} \subset B\mathrm{O}(2)$.]

Let R stand for the subring generated by x_1 and x_2, so that there is an additive splitting

$$H^*(B(\mathrm{P}^m, 2); \mathbb{F}_2) = R \oplus x \cdot R \tag{36}$$

which is compatible with the action of Sq^1 (note that multiplication by x determines an additive isomorphism $R \cong x \cdot R$).

Proposition 6.4. *Let $m = 4a+3$. With respect to the differential* Sq^1 :

- $H^{m+1}(R; \mathrm{Sq}^1) = \mathbb{Z}_2$.
- $H^{m+1}(x \cdot R; \mathrm{Sq}^1) = 0$.

Before proving this result, let us indicate how it can be used to show that (20) is an isomorphism for $m = 4a+3$. As explained in the paragraph containing (34), we must have

$$2 \cdot H^{4a+4}(B(\mathrm{P}^{4a+3}, 2)) = \langle r \rangle \quad \text{with} \quad r \geq 1 \tag{37}$$

and we need to show that $r = 1$ is in fact the case. Consider the Bockstein exact couple

$$\begin{tikzcd}[column sep=small]
H^*(B(P^{4a+3},2)) \arrow[rr, "2"] & & H^*(B(P^{4a+3},2)) \arrow[dl, "\rho"] \\
& H^*(B(P^{4a+3},2);\mathbb{F}_2). \arrow[ul, "\delta"] &
\end{tikzcd}$$

In the (unravelled) derived exact couple

$$\cdots \to 2 \cdot H^{4a+4}(B(P^{4a+3}, 2)) \xrightarrow{2} 2 \cdot H^{4a+4}(B(P^{4a+3}, 2)) \to$$
$$\to H^{4a+4}(H^*(B(P^{4a+3}, 2); \mathbb{F}_2); \mathrm{Sq}^1) \to 2 \cdot H^{4a+5}(B(P^{4a+3}, 2)) \to \cdots$$

we have $2 \cdot H^{4a+5}(B(P^{4a+3}, 2)) = 0$ since $H^{4a+5}(B(P^{4a+3}, 2)) = \langle 2a+1 \rangle$—argued in Section 4 by means of the (twisted) torsion linking form. Together with (37), this implies that the map

$$\langle r \rangle = 2 \cdot H^{4a+4}(B(P^{4a+3}, 2)) \to H^{4a+4}(H^*(B(P^{4a+3}, 2); \mathbb{F}_2); \mathrm{Sq}^1) \tag{38}$$

in the above exact sequence is an isomorphism. Proposition 6.4 and (36) then imply the required conclusion $r = 1$.

Proof of Proposition 6.4. Note that every binomial coefficient in (II) with $i \not\equiv 0 \bmod 4$ is congruent to zero mod 2. Therefore relation (II) can be rewritten as

$$x_1^{4a+3} = \sum_{j=1}^{a/2} \binom{a-j}{j} x_1^{4(a-2j)+3} x_2^{4j}. \tag{39}$$

Likewise, every binomial coefficient in (III) with $i \equiv 3 \bmod 4$ is congruent to zero mod 2. Then, taking into account (39), relation (III) becomes

$$\begin{aligned} x_2^{2a+2} &= x_1^{4a+4} + \sum_{i\in\Lambda} \binom{4a+4-i}{i} x_1^{4a+4-2i} x_2^i \\ &= \sum_{j=1}^{a/2} \binom{a-j}{j} x_1^{4(a-2j)+4} x_2^{4j} + \sum_{i\in\Lambda} \binom{4a+4-i}{i} x_1^{4a+4-2i} x_2^i \end{aligned} \tag{40}$$

where Λ is the set of integers i with $1 \le i \le 2a+1$ and $i \not\equiv 3 \bmod 4$. Using (39) and (40) it is a simple matter to write down a basis for R and $x \cdot R$ in dimensions $4a+3$, $4a+4$, and $4a+5$. The information is summarized (under the assumption $a > 0$, which is no real restriction in view of (34)) in the following chart, where elements in a column form a basis in the indicated dimension, and where crossed out terms can be expressed as linear combination of the other ones in view of (39) and (40).

$$\begin{array}{lllll}
4a+3 & & 4a+4 & & 4a+5 \\
\cancel{x_1^{4a+3}} & & \cancel{x_1^{4a+4}} & & \cancel{x_1^{4a+5}} \\
x_1^{4a+1}x_2 & \to 0 & x_1^{4a+2}x_2 & & \cancel{x_1^{4a+3}x_2} \\
x_1^{4a-1}x_2^2 & \longrightarrow & x_1^{4a}x_2^2 & & x_1^{4a+1}x_2^2 \\
x_1^{4a-3}x_2^3 & \to 0 & x_1^{4a-2}x_2^3 & \longrightarrow & x_1^{4a-1}x_2^3 \\
\vdots & & \vdots & & x_1^{4a-3}x_2^4 \\
 & & & & \vdots \\
x_1^3x_2^{2a} & \longrightarrow & x_1^4x_2^{2a} & & \\
x_1x_2^{2a+1} & \to 0 & x_1^2x_2^{2a+1} & \longrightarrow & x_1^3x_2^{2a+1} \\
 & & \cancel{x_2^{2a+2}} & & \cancel{x_1x_2^{2a+2}} \\
\hline
xx_1^{4a+2} & & \cancel{xx_1^{4a+3}} & & \cancel{xx_1^{4a+4}} \\
xx_1^{4a}x_2 & \to 0 & xx_1^{4a+1}x_2 & \longrightarrow & xx_1^{4a+2}x_2 \\
xx_1^{4a-2}x_2^2 & \longrightarrow & xx_1^{4a-1}x_2^2 & & xx_1^{4a}x_2^2 \\
 & & xx_1^{4a-3}x_2^3 & \longrightarrow & xx_1^{4a-2}x_2^3 \\
\vdots & & \vdots & & \vdots \\
xx_1^2x_2^{2a} & \longrightarrow & xx_1^3x_2^{2a} & & \\
xx_2^{2a+1} & \to 0 & xx_1x_2^{2a+1} & \longrightarrow & xx_1^2x_2^{2a+1} \\
 & & & & \cancel{xx_2^{2a+2}}
\end{array}$$

The top and bottom portions of the chart (delimited by the horizontal dotted line) correspond to R and $x \cdot R$, respectively. Horizontal arrows indicate Sq^1-images, which are easily computable from (35) and (I): $\mathrm{Sq}^1(x^i x_1^{i_1} x_2^{i_2}) = 0$ when $i + i_1 + i_2$ is even, while $\mathrm{Sq}^1(x^i x_1^{i_1} x_2^{i_2}) = x^i x_1^{i_1+1} x_2^{i_2}$ when $i + i_1 + i_2$ is odd—here $i \in \{0, 1\}$ in view of (I) above. There are only two basis elements, in dimensions $4a+3$ and $4a+4$, whose Sq^1-images are not indicated in the chart: $xx_1^{4a+2} \in (x \cdot R)^{4a+3}$ and $x_1^{4a+2} x_2 \in R^{4a+4}$. The second conclusion in the proposition is evident from the bottom part of the chart—no matter what the Sq^1-image of xx_1^{4a+2} is. On the other hand, the top portion of the chart implies that, in dimension $4a+4$, $\mathrm{Ker}(\mathrm{Sq}^1)$ and $\mathrm{Im}(\mathrm{Sq}^1)$ are elementary 2-groups whose ranks satisfy

$$\mathrm{rk}(\mathrm{Ker}(\mathrm{Sq}^1)) = \mathrm{rk}(\mathrm{Im}(\mathrm{Sq}^1)) + \varepsilon$$

with $\varepsilon = 1$ or $\varepsilon = 0$ (depending on whether or not $\mathrm{Sq}^1(x_1^{4a+2} x_2)$ can be written down as a linear combination of the elements $x_1^{4a-1} x_2^3$, $x_1^{4a-5} x_2^5, \ldots,$ and $x_1^3 x_2^{2a+1}$—this of course depends on the actual binomial coefficients in (39)). But the possibility $\varepsilon = 0$ is ruled out by (37) and (38), forcing $\varepsilon = 1$ and, therefore, the first assertion of this proposition. $\square$

7 Case of $F(\mathrm{P}^m, 2)$

The CLSS analysis in the previous two sections can be applied—with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $G = D_8$—in order to study the cohomology groups of the ordered configuration space $F(\mathrm{P}^m, 2)$. The explicit details are similar but much easier than those for unordered configuration spaces, and this time the additive structure of differentials can be fully understood for any m. Here we only review the main differences, simplifications, and results.

For one, there is no 4-torsion to deal with (e.g. the arithmetic Proposition 5.2 is not needed); indeed, the role of BD_8 in the situation of an unordered configuration space $B(\mathrm{P}^m, 2)$ is played by $\mathrm{P}^\infty \times \mathrm{P}^\infty$ for ordered configuration spaces $F(\mathrm{P}^m, 2)$. Thus, the use of Corollaries 2.4 and 2.5 is replaced by the simpler Lemma 2.8. But the most important simplification in the calculations relevant to the present section comes from the absence of problematic d_2-differentials, the obstacle that prevented us from computing the CLSS of the D_8-action on $V_{m+1,2}$ for $m \equiv 3 \bmod 4$. [This is why in Lemma 2.8 we do not insist on describing $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{Z}_\alpha)$ as a module over $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty)$—compare to Remark 5.6.] As a result, the integral cohomology CLSS of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on $V_{m+1,2}$ can be fully understood, without restriction on m, by means of the counting arguments used in Section 5, now forcing the injectivity of all relevant differentials from the following two ingredients:

(a) The size and distribution of the groups in the CLSS.

(b) The $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of Proposition 3.2 in Remark 3.3—the input triggering the determination of differentials.

In particular, when m is odd, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of Lemma 5.5 does not arise and, instead, only the counting argument in the proof following Remark 5.6 is needed.

We leave it for the reader to supply details of the above CLSS and verify that this leads to Theorems 1.9 and 1.10 in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, as well as to the computation of all the cohomology groups in Theorem 1.1.

8 The symmetric topological complexity of P^5 and P^6

In this final section we use the cohomological information gathered in previous sections in order to compute the symmetric topological complexity of P^m for $m = 5, 6$ (Theorem 1.4). The method is an extension of that used in [17] to deal with the case $m = 3$.

Definition 8.1. The topological complexity of a space X, $\mathrm{TC}(X)$, is defined as the *reduced* Schwarz genus of the endpoints evaluation map $\mathrm{ev}\colon X^{[0,1]} \to X \times X$, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, i.e. $\mathrm{TC}^S(X) + 1$ gives the smallest cardinality of covers of $X \times X$ by open sets over each of which ev admits a (continuous) section. To define a symmetric version of $\mathrm{TC}(X)$, note that the involution on $X \times X$ that switches coordinates is compatible, via ev, with the involution on $X^{[0,1]}$ that reverses a path. These actions are free on the domain and codomain of the restricted fibration $\mathrm{ev}\colon \mathrm{ev}^{-1}(F(X,2)) \to F(X,2)$ which thus, at the level of orbit spaces, yields a fibration $\mathrm{ev}'\colon \mathrm{ev}^{-1}(F(X,2))/\mathbb{Z}_2 \to B(X,2)$. The symmetric topological complexity of X, $\mathrm{TC}^S(X)$, is defined to be one less than the reduced Schwarz genus of ev'.

Remark 8.2. The adjustment by one in the definition of $\mathrm{TC}^S(X)$ does not come from any normalization convention—it can be thought of as accounting for the obvious symmetric section of ev over the (removed) diagonal. Instead, the normalization we have taken for the Schwarz genus means that, just as in [17], the values of $\mathrm{TC}(X)$ and $\mathrm{TC}^S(X)$ in this paper are one less than those originally defined in [10, 11].

Before getting into the main technical computation of this section, it is convenient to set Theorem 1.4 in context. The inequality

$$\mathrm{TC}^S(X) - \mathrm{TC}(X) \geq 0 \tag{41}$$

is proved in [11, Corollary 9] for any space X. It is optimal since, as proved in [18], (41) becomes an equality when X is, for instance, a complex projective space. However, as discussed in [18, Example 3.3], there is no current indication that the left hand side in (41) should even be a bounded function of m for $X = \mathrm{P}^m$. We discuss the known situation (as updated by Theorem 1.4) for a few particular families of m. In the following paragraph we use [7, 12] as the main references for the known numerical values of $\mathrm{TC}(\mathrm{P}^m)$.

To begin with, Example 3.3 in [18] observes that

$$\mathrm{TC}^S(\mathrm{P}^{2^i}) - \mathrm{TC}(\mathrm{P}^{2^i}) = 1$$

for any $i \geq 0$ (the case $i = 0$ was not mentioned in [18], but it is covered by the calculations in [10, 11]). Example 3.3 in [18] also notes that

$$\mathrm{TC}^S(\mathrm{P}^{2^i+1}) - \mathrm{TC}(\mathrm{P}^{2^i+1}) = 2$$

for any $i \geq 3$; the corresponding result for $i = 1, 2$ is also true in view of [17] (for $i = 1$) and Theorem 1.4 (for $i = 2$). Lastly, Example 3.3 in [18] remarks that

$$\mathrm{TC}^S(\mathrm{P}^{2^i+2}) - \mathrm{TC}(\mathrm{P}^{2^i+2}) = 1 \tag{42}$$

for any $i \geq 4$. Now, while (42) is also true for $i = 3$ (as remarked in [17, Table 1]), Theorem 1.4 implies that, for $i = 2$, (42) must be replaced by $\mathrm{TC}^S(\mathrm{P}^6) - \mathrm{TC}(\mathrm{P}^6) = 2$.

We now start working toward the proof of Theorem 1.4. As recalled in the Introduction, for any $m \geq 1$, $\mathrm{TC}^S(\mathrm{P}^m)$ agrees with the smallest positive integer $n = n(m)$ for which the map in (1) can be homotopy compressed into P^{n-1}. We take advantage of the obvious inequality $n(m) \leq n(m+1)$: since $n(6) \leq 9$ ([37, Corollary 11]), Theorem 1.4 will follow once we show that the case $m = 5$ of the map u in (1) cannot be homotopy compressed into P^7. We prove in fact:

Theorem 8.3. *The nonzero element $z \in H^2(\mathrm{P}^\infty)$ satisfies $u^*(z)^4 \neq 0$.*

Our proof of Theorem 8.3 is based on a direct study of the CLSS for the $\mathbb{Z}_2$-action on $F(\mathrm{P}^5, 2)$ in Definition 8.1 which, by definition, is classified by u. So, our first goal—accomplished in Proposition 8.6 below—is to describe the (highly) twisted coefficients of this spectral sequence, i.e. the action in integral cohomology of the involution on $F(\mathrm{P}^5, 2)$ that switches coordinates.

The $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on $V_{m+1,2}$ given in Definition 1.6 extends to the standard product action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $S^\infty \times S^\infty$. Thus the sequence of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant inclusions $V_{m+1,2} \hookrightarrow S^m \times S^m \hookrightarrow S^\infty \times S^\infty$ shows that the map $p = p_{m,\mathbb{Z}_2\times\mathbb{Z}_2} \colon F(\mathrm{P}^m, 2) \to \mathrm{P}^\infty \times \mathrm{P}^\infty$ in (4) factors (up to homotopy) as

$$F(\mathrm{P}^m, 2) \hookrightarrow \mathrm{P}^m \times \mathrm{P}^m \hookrightarrow \mathrm{P}^\infty \times \mathrm{P}^\infty. \tag{43}$$

This fact is used in the proof of the following mod 2 result, which was brought to the authors' attention by Fred Cohen. Recall the cohomology classes $x_1, y_1 \in H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{F}_2)$ and $x_2, y_2, z_3 \in H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty)$ introduced in the paragraph containing (6).

Lemma 8.4. *The morphism* $p^*\colon H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty;\mathbb{F}_2) \to H^*(F(\mathrm{P}^m,2);\mathbb{F}_2)$ *is surjective with kernel the ideal generated by the three elements* x_1^{m+1}*,* y_1^{m+1}*, and* $\sum_{i+j=m} x_1^i y_1^j$*.*

Proof. The first two elements generate the kernel of the second inclusion in (43), whereas the third element maps to the diagonal cohomology class in $\mathrm{P}^m \times \mathrm{P}^m$ in view of [33, Theorem 11.11]—which certainly restricts to zero in $F(\mathrm{P}^m,2)$. So it suffices to check that the first inclusion in (43) is surjective with kernel generated by the diagonal class. But [33, Section 11] embeds the map under consideration into a long exact sequence

$$\cdots \to H^{*-m}(P^m;\mathbb{Z}_2) \to H^*(\mathrm{P}^m \times \mathrm{P}^m;\mathbb{Z}_2) \to H^*(F(\mathrm{P}^m,2);\mathbb{Z}_2) \to \cdots$$

(written here in terms of the Thom isomorphism for the normal bundle of the diagonal inclusion $\mathrm{P}^m \hookrightarrow \mathrm{P}^m \times \mathrm{P}^m$). The desired conclusion then follows from [33, Lemma 11.8] which shows that the map of degree m in this long exact sequence is given by multiplication by the diagonal class $\sum_{i+j=m} x_1^i y_1^j$—clearly a monomorphism in the current case. □

The argument in the previous proof cannot be applied with integer coefficients for a non-orientable projective space (or manifold, for that matter). Nevertheless we prove:

Corollary 8.5. *The kernel of* $p^*\colon H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty) \to H^*(F(\mathrm{P}^5,2))$ *is the ideal generated by the three elements* x_2^3*,* y_2^3*, and* $z_3(x_2^2 + x_2y_2 + y_2^2)$*. Further, an* $\mathbb{F}_2$*-basis for the torsion groups in* $H^*(F(\mathrm{P}^5,2))$ *is given by the (*p^**-images of the) elements in Table 2.*

$*=$	0	1	2	3	4	5	6	7
	—	—	$x_2,\ y_2$	z_3	$x_2^2,\ x_2y_2,\ y_2^2$	$x_2z_3,\ y_2z_3$	$x_2^2y_2,\ x_2y_2^2$	$x_2^2z_3,\ y_2^2z_3$

Table 2: Basis elements for $TH^*(F(\mathrm{P}^5,2))$ through $*\leq 7$

Proof. The P^5-case of Theorem 1.1 implies that the mod 2 reduction map $H^*(F(\mathrm{P}^5,2)) \to H^*(F(\mathrm{P}^5,2);\mathbb{F}_2)$ is injective in positive dimensions not 5, so that a straightforward calculation using Remark 2.9 and Lemma 8.4 yields that the three indicated classes lie in the kernel of p^*. The result then follows from an easy counting argument, taking into account (6), Theorem 1.10 (for $m = 5$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$), and the full description of $H^*(F(\mathrm{P}^5,2))$ in Theorem 1.1. □

As suggested by Corollary 8.5, it will be convenient to denote elements in the torsion groups of $H^*(F(\mathrm{P}^5,2))$ by their corresponding preimages in Table 2. Next we choose a generator of the torsion-free summand in $H^5(F(\mathrm{P}^5,2))$. Since $H^6(F(\mathrm{P}^5,2))$ is an $\mathbb{F}_2$-vector

space, the image of the reduction map $H^5(F(\mathrm{P}^5,2)) \to H^5(F(\mathrm{P}^5,2);\mathbb{F}_2)$ agrees with the kernel of $\mathrm{Sq}^1\colon H^5(F(\mathrm{P}^5,2);\mathbb{F}_2) \to H^6(F(\mathrm{P}^5,2);\mathbb{F}_2)$. The latter is easily seen to have

$$x_1^3y_1(x_1+y_1),\ \ x_1y_1^3(x_1+y_1),\ \text{ and } x_1^5 \tag{44}$$

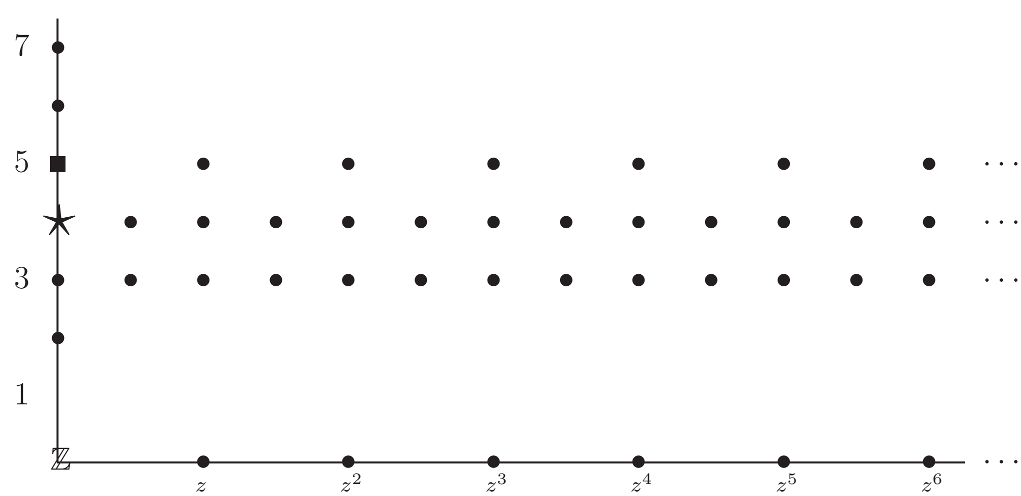
as an $\mathbb{F}_2$-basis (although y_1^5 is in the kernel of Sq^1, it is not a new basis element because of the relation coming from the third element in Lemma 8.4). Now, the first two elements in (44) are the corresponding mod 2 reductions of the two basis elements noted in dimension 5 in Table 2. Therefore the torsion-free summand in $H^5(F(\mathrm{P}^5,2))$ is generated by a class w_5 having x_1^5 as its mod 2 reduction.

Proposition 8.6. *The automorphism induced in $\mathbb{Z}$-cohomology by the involution ρ that switches coordinates in $F(\mathrm{P}^5,2)$ is characterized by*

$$\rho^*(x_2)=y_2,\quad \rho^*(z_3)=z_3,\quad \textit{and}\quad \rho^*(w_5)=w_5+(x_2+y_2)z_3. \tag{45}$$

Proof. Note that the fibration $V_{6,2} \xrightarrow{\theta} F(\mathrm{P}^5,2) \xrightarrow{p} \mathrm{P}^\infty \times \mathrm{P}^\infty$ is ρ-equivariant. The first equality in (45) is obvious since x_2 and y_2 ultimately come from the axes in $\mathrm{P}^\infty\times\mathrm{P}^\infty$. The second equality is forced since $H^3(\mathrm{P}^5,2)=\mathbb{Z}_2$. For the third equality we necessarily have $\rho^*(w_5)=\varepsilon w_5+\delta_1x_2z_3+\delta_2y_2z_3$ with $\varepsilon=\pm1$ and $\delta_i\in\{0,1\}$. Since w_5 maps nontrivially under the fiber inclusion of p, Theorem 3.5 forces $\varepsilon=1$. The fact that $\delta_1=\delta_2=1$ then follows easily by reducing coefficients modulo 2 and using the relation coming from the third generator in Lemma 8.4. □

The $E_2^{p,q}$-term in the CLSS of the involution ρ in Proposition 8.6 can now be obtained from standard calculations. The result, recorded in Corollary 8.7 below, is depicted in the following chart for $q\le 7$, where a bullet (respectively square, star) stands for a copy of $\mathbb{Z}_2$ (respectively $\mathbb{Z}\oplus\mathbb{Z}_2$, $\mathbb{Z}_2\oplus\mathbb{Z}_2$).



Corollary 8.7. *Let H^i stand for $H^i(F(\mathrm{P}^5, 2))$ as a $\mathbb{Z}_2$-module via the action of ρ^*. Then:*

1. $H^0 = \mathbb{Z}$, *a trivial $\mathbb{Z}_2$-module, so* $H^*(\mathrm{P}^\infty; H^0) = \mathbb{Z}[z]/2z$, $\deg(z) = 2$.

2. $H^1 = 0$, *so* $H^*(\mathrm{P}^\infty; H^1) = 0$.

3. *For* $i = 2, 6, 7$, $H^i = \mathbb{Z}_2[\mathbb{Z}_2]$, *so* $H^*(\mathrm{P}^\infty; H^i) = \mathbb{Z}_2$ *concentrated in degree* 0.

4. $H^3 = \mathbb{Z}_2$, *so* $H^*(\mathrm{P}^\infty; H^3) = \mathbb{Z}_2[x]$, $\deg(x) = 1$.

5. $H^4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2[\mathbb{Z}_2]$, *so* $H^*(\mathrm{P}^\infty; H^4) = \mathbb{Z}_2[x] \oplus \mathbb{Z}_2$ *where the second summand is concentrated in degree* 0.

6. $H^5 = \mathbb{Z} \oplus \mathbb{Z}_2[\mathbb{Z}_2]$ *where the additive (torsion) subgroup is in fact a $\mathbb{Z}_2$-submodule, but* $\rho^*(w_5) = w_5 + (1 + \rho^*)g$ *(w_5 generates $\mathbb{Z}$, and g generates the $\mathbb{Z}_2$-module $\mathbb{Z}_2[\mathbb{Z}_2]$), so*

$$H^*(\mathrm{P}^\infty; H^5) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & * = 0; \\ \mathbb{Z}_2, & * = 2a, \ a > 0; \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 8.8. With respect to the multiplicative structure of the CLSS, a standard cohomology calculation gives that $z \in E_2^{2,0}$ acts injectively on $H^*(\mathrm{P}^\infty; H^3)$, on positive dimensions of $H^*(\mathrm{P}^\infty; H^4)$ and $H^*(\mathrm{P}^\infty; H^5)$, and on the torsion subgroup of $E_2^{0,5} = \mathbb{Z} \oplus \mathbb{Z}_2$. Furthermore, for the purpose of the CLSS analysis in the proof of Theorem 8.3, we will *choose* a non-torsion generator in $E_2^{0,5}$ so that all of its z^i-multiples are nonzero.

Proof of Theorem 8.3. The generator of $E_2^{0,3}$ must be a permanent cycle since, in view of Theorem 1.2, $H^3(B(\mathrm{P}^5, 2)) = \mathbb{Z}_2$—in the sequel we will refer to this sort of argument as "by convergence". Since there is no nontrivial target for the d_4-differential on the generator of $E_2^{1,3}$, the multiplicative structure of the spectral sequence (Remark 8.8) shows that the whole $(q = 3)$-line consists of permanent cycles. This leaves three differentials, originating at nodes

$$(3, 4), \quad (2, 5), \quad \text{and} \quad (0, 7), \tag{46}$$

possibly hitting $z^4 \in E_2^{8,0}$. The proof will be complete once we show that z^4 is not hit by any of these differentials.

By convergence, all of $E_2^{0,4}$ consists of permanent cycles. One of these elements is given by the ρ^*-invariant element $x_2^2 + y_2^2$ (see Table 2). Since the permanent cycle in $E_2^{0,3}$ is given by the ρ^*-invariant element z_3, the product $(x_2^2 + y_2^2)z_3$—giving the generator of $E_2^{0,7}$—is a permanent cycle too. This accounts for the d_8-differential in (46). A second conclusion we draw at this point is that the survival of all of $E_2^{0,4}$ in the spectral sequence implies (in

view Remark 8.8) that elements of even total degree in the $(q = 4)$-line are also permanent cycles.

Before analyzing the two remaining differentials potentially hitting z^4, we deduce a few more permanent cycles in the spectral sequence. Firstly, we have observed that z_3 gives the generator in $E_2^{0,3}$; now the third relation in (6) shows that the generator in $E_2^{0,6}$ is a permanent cycle. Secondly, since $x_2 + y_2$—the generator in $E_2^{0,2}$—is a permanent cycle (say by convergence), $(x_2 + y_2)z_3$—the generator of the torsion element in $E_2^{0,5}$—is another permanent cycle. Lastly, Remark 8.8 implies that all torsion elements in the $(q = 5)$-line are also permanent cycles. Of course, the last assertion accounts for the d_6-differential in (46).

So far we have proved that, in the range shown in the chart, the only elements potentially supporting a nonzero differential are (a) the torsion-free generator in $E_2^{0,5}$—chosen in Remark 8.8—and the elements in the $(q = 4)$-line having odd total degree. We next argue that there must be a nonzero d_k-differential (with $k \in \{2, 5\}$) originating at node $(1, 4)$. Indeed, at the start of the spectral sequence there are three nonzero homogeneous torsion elements in total degree 5, however by convergence there are only two such elements in the E_∞-term; the extra element must be the source of a nonzero differential (recall that $E_2^{0,4}$ consists of permanent cycles). But our analysis of permanent cycles shows that such a differential can originate only at node $(1, 4)$, as asserted. Now, if the d_2-differential originating at node $(1, 4)$ is the one that is nonzero, then Remark 8.8 implies that this differential repeats horizontally every two degrees, killing in particular the element at node $(3, 4)$ and, therefore, accounting for the remaining d_5-differential in (46).

The proof is concluded by drawing a contradiction from the assumption that the d_2-differential originating at node $(1, 4)$ vanishes. Indeed, such an hypothesis, Remark 8.8 and our analysis of permanent cycles would imply, on the one hand, that all d_2-differentials originating at the $(q = 4)$-line must vanish and, on the other, that the torsion-free generator in $E_2^{0,5}$ (chosen in Remark 8.8) is a d_ℓ-cycle for $\ell = 2, 3$. In turn, this situation would imply that the permanent cycles at nodes $(3, 3)$ and $(2, 4)$ are not killed by any differential. Since this is also the case for the permanent cycle at node $(0, 6)$, we would have identified three nonzero torsion homogeneous elements in total degree 6 in the E_∞-term. But this is impossible by convergence. □

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Morfismos, Vol. 14, No. 2, 2010, pp. 17–59

Cohomology groups of configuration spaces of pairs of points in real projective spaces

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Abstract

The Stiefel manifold $V_{m+1,2}$ of 2-frames in $\mathbb{R}^{m+1}$ is acted upon by the orthogonal group $\mathrm{O}(2)$. By restriction, there are corresponding actions of the dihedral group of order 8, D_8, and of the rank-2 elementary 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We use the Cartan-Leray spectral sequences of these actions to compute the integral homology and cohomology groups of the configuration spaces $B(\mathrm{P}^m, 2)$ and $F(\mathrm{P}^m, 2)$ of (unordered and ordered) pairs of points on the real projective space P^m.

2010 Mathematics Subject Classification: 55R80, 55T10, 55M30, 57R19, 57R40.
Keywords and phrases: 2-point configuration spaces, dihedral group of order 8, twisted Poincaré duality, torsion linking form.

1 Introduction

The integral cohomology rings of the configuration spaces $F(\mathrm{P}^m, 2)$ and $B(\mathrm{P}^m, 2)$ of two distinct points, ordered and unordered respectively, in the m-dimensional real projective space P^m have recently been computed in [6]. The method in that paper relies on a rather technical bookkeeping in the corresponding Bockstein spectral sequences. As a consequence, a reader following the details in that work might miss part of the geometrical insight of the problem (in Definition 1.4 and subsequent considerations). To help remedy such a situation, we offer in this paper an alternative approach to the additive structure.

[1]Partially supported by CONACYT Research Grant 102783.

The basic results are presented in Theorems 1.1 and 1.2 below, where the notation $\langle k\rangle$ stands for the elementary abelian 2-group of rank k, $\mathbb{Z}_2\oplus\cdots\oplus\mathbb{Z}_2$ (k times), and where we write $\{k\}$ as a shorthand for $\langle k\rangle\oplus\mathbb{Z}_4$.

Theorem 1.1. *For $n>0$,*

$$H^i(F(\mathrm{P}^{2n},2))=\begin{cases}\mathbb{Z}, & i=0 \quad or \quad i=4n-1;\\ \left\langle \frac{i}{2}+1\right\rangle, & i \quad even,\;\; 1\le i\le 2n;\\ \left\langle \frac{i-1}{2}\right\rangle, & i \quad odd,\;\; 1\le i\le 2n;\\ \left\langle 2n+1-\frac{i}{2}\right\rangle, & i \quad even,\;\; 2n<i<4n-1;\\ \left\langle 2n-\frac{i+1}{2}\right\rangle, & i \quad odd,\;\; 2n<i<4n-1;\\ 0, & otherwise.\end{cases}$$

For $n\ge 0$,

$$H^i(F(\mathrm{P}^{2n+1},2))=\begin{cases}\mathbb{Z}, & i=0;\\ \left\langle \frac{i}{2}+1\right\rangle, & i \quad even,\;\; 1\le i\le 2n;\\ \left\langle \frac{i-1}{2}\right\rangle, & i \quad odd,\;\; 1\le i\le 2n;\\ \mathbb{Z}\oplus\langle n\rangle, & i=2n+1;\\ \left\langle 2n+1-\frac{i}{2}\right\rangle, & i \quad even,\;\; 2n+1<i\le 4n+1;\\ \left\langle 2n+1-\frac{i-1}{2}\right\rangle, & i \quad odd,\;\; 2n+1<i\le 4n+1;\\ 0, & otherwise.\end{cases}$$

Theorem 1.2. *Let $0\le b\le 3$. For $n>0$,*

$$H^{4a+b}(B(\mathrm{P}^{2n},2))=\begin{cases}\mathbb{Z}, & 4a+b=0 \;\; or \;\; 4a+b=4n-1;\\ \{2a\}, & b=0<a,\;\; 4a+b\le 2n;\\ \langle 2a\rangle, & b=1,\;\; 4a+b\le 2n;\\ \langle 2a+2\rangle, & b=2,\;\; 4a+b\le 2n;\\ \langle 2a+1\rangle, & b=3,\;\; 4a+b\le 2n;\\ \{2n-2a\}, & b=0,\;\; 2n<4a+b<4n-1;\\ \langle 2n-2a-1\rangle, & b=1,\;\; 2n<4a+b<4n-1;\\ \langle 2n-2a\rangle, & b=2,\;\; 2n<4a+b<4n-1;\\ \langle 2n-2a-2\rangle, & b=3,\;\; 2n<4a+b<4n-1;\\ 0, & otherwise.\end{cases}$$

For $n \geq 0$,

$$H^{4a+b}(B(\mathrm{P}^{2n+1},2)) = \begin{cases} \mathbb{Z}, & 4a+b=0; \\ \{2a\}, & b=0<a,\ \ 4a+b<2n+1; \\ \langle 2a\rangle\,, & b=1,\ \ 4a+b<2n+1; \\ \langle 2a+2\rangle\,, & b=2,\ \ 4a+b<2n+1; \\ \langle 2a+1\rangle\,, & b=3,\ \ 4a+b<2n+1; \\ \mathbb{Z}\oplus\langle n\rangle, & 4a+b=2n+1; \\ \{2n-2a\}, & b=0,\ \ 2n+1<4a+b\leq 4n+1; \\ \langle 2n+1-2a\rangle, & b=1,\ \ 2n+1<4a+b\leq 4n+1; \\ \langle 2n-2a\rangle, & b\in\{2,3\},\ \ 2n+1<4a+b\leq 4n+1; \\ 0, & \textit{otherwise}. \end{cases}$$

As noted in [6], Theorems 1.1 and 1.2 can be coupled with the Universal Coefficient Theorem (UCT), expressing homology in terms of cohomology (e.g. [22, Theorem 56.1]), in order to give explicit descriptions of the corresponding integral homology groups. Another immediate consequence is that, together with Poincaré duality (in its not necessarily orientable version, cf. [17, Theorem 3H.6] or [24, Theorem 4.51]), Theorems 1.1 and 1.2 give a corresponding explicit description of the w_1-twisted homology and cohomology groups of $F(\mathrm{P}^m,2)$ and $B(\mathrm{P}^m,2)$. Details are given in Section 4—a second contribution not discussed in [6].

Remark 1.3. Note that, after inverting 2, both $B(\mathrm{P}^m,2)$ and $F(\mathrm{P}^m,2)$ are homology spheres. This assertion can be considered as a partial generalization of the fact that both $F(\mathrm{P}^1,2)$ and $B(\mathrm{P}^1,2)$ have the homotopy type of a circle; for $B(\mathrm{P}^1,2)$ this follows from Lemma 1.6 and Example 3.4 below, while the situation for $F(\mathrm{P}^1,2)$ comes from the fact that P^1 is a Lie group—so that $F(\mathrm{P}^1,2)$ is in fact diffeomorphic to $S^1\times(S^1-\{1\})$. In particular, any product of positive dimensional classes in either $H^*(F(\mathrm{P}^1,2))$ or $H^*(B(\mathrm{P}^1,2))$ is trivial. The trivial-product property also holds for both $H^*(F(\mathrm{P}^2,2))$ and $H^*(B(\mathrm{P}^2,2))$ in view of the P^2-case in Theorems 1.1 and 1.2. For $m\geq 3$, the multiplicative structure of $H^*(F(\mathrm{P}^m,2))$ and $H^*(B(\mathrm{P}^m,2))$ was first worked out in [5].

Definition 1.4. Recall that D_8 can be expressed as the usual wreath product extension

$$1\to\mathbb{Z}_2\times\mathbb{Z}_2\to D_8\to\mathbb{Z}_2\to 1. \tag{1}$$

Let $\rho_1,\rho_2\in D_8$ generate the normal subgroup $\mathbb{Z}_2\times\mathbb{Z}_2$, and let (the class of) $\rho\in D_8$ generate the quotient group $\mathbb{Z}_2$ so that, via conjugation,

ρ switches ρ_1 and ρ_2. D_8 acts freely on the Stiefel manifold $V_{n,2}$ of orthonormal 2-frames in $\mathbb{R}^n$ by setting

$$\rho(v_1, v_2) = (v_2, v_1), \quad \rho_1(v_1, v_2) = (-v_1, v_2), \quad \text{and} \quad \rho_2(v_1, v_2) = (v_1, -v_2).$$

This describes a group inclusion $D_8 \hookrightarrow \mathrm{O}(2)$ where the rotation $\rho\rho_1$ is a generator for $\mathbb{Z}_4 = D_8 \cap \mathrm{SO}(2)$.

Notation 1.5. Throughout the paper the letter G stands for either D_8 or its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (1). Likewise, $E_m = E_{m,G}$ denotes the orbit space of the G-action on $V_{m+1,2}$ indicated in Definition 1.4, and $\theta\colon V_{m+1,2} \to E_{m,G}$ represents the canonical projection. Our interest lies in the (kernel of the) morphism induced in cohomology by the map

$$p = p_{m,G} \colon E_m \to BG \tag{2}$$

that classifies the G-action on $V_{m+1,2}$.

Lemma 1.6 ([15, Proposition 2.6])**.** *E_m is a strong deformation retract of $B(\mathrm{P}^m, 2)$ if $G = D_8$, and of $F(\mathrm{P}^m, 2)$ if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.* □

Thus, the cohomology properties of the configuration spaces we are interested in—and of (2), for that matter—can be approached via the Cartan-Leray spectral sequence (CLSS) of the G-action on $V_{m+1,2}$. Such an analysis yields:

Proposition 1.7. *Let m be even. The map $p^*\colon H^i(BG) \to H^i(E_m)$ is:*

1. *an isomorphism for $i \leq m$;*
2. *an epimorphism with nonzero kernel for $m < i < 2m-1$;*
3. *the zero map for $2m-1 \leq i$.*

Proposition 1.8. *Let m be odd. The map $p^*\colon H^i(BG) \to H^i(E_m)$ is:*

1. *an isomorphism for $i < m$;*
2. *a monomorphism onto the torsion subgroup of $H^i(E_m)$ for $i = m$;*
3. *an epimorphism with nonzero kernel for $m < i \leq 2m-1$.*
4. *the zero map for $2m-1 < i$.*

Kernels in the above two results are carefully described in [6]. The approach in this paper allows us to prove Propositions 1.7 and 1.8, except for item 3 in Proposition 1.8 if $G = D_8$ and $m \equiv 3 \bmod 4$.

Since the ring $H^*(BG)$ is well known (see Theorem 2.3 and the comments following Lemma 2.8), the multiplicative structure of $H^*(E_m)$ through dimensions at most m follows from the four results stated in this section. Of course, the ring structure in larger dimensions depends on giving explicit generators for the ideal $\mathrm{Ker}(p^*)$. In this direction we note that the methods in this paper also yield:

Proposition 1.9. *Let $G = D_8$. Assume $m \not\equiv 3 \bmod 4$ and consider the map in (2). In dimensions at most $2m-1$, every nonzero element in $\mathrm{Ker}(p^*)$ has order 2, i.e. $2 \cdot \mathrm{Ker}(p^*) = 0$ in those dimensions. In fact, every 4ℓ-dimensional integral cohomology class in BD_8 generating a $\mathbb{Z}_4$-group maps under p^* into a class which also generates a $\mathbb{Z}_4$-group provided $\ell < m/2$—otherwise the class maps trivially for dimensional reasons.*

Remark 1.10. By Lemma 2.8 below, $\mathrm{Ker}(p^*)$ is also killed by multiplication by 2 when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (any m, any dimension). Our approach allows us to explicitly describe the (dimension-wise) 2-rank of $\mathrm{Ker}(p^*)$ in the cases where we know this is an $\mathbb{F}_2$-vector space (i.e. when either $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $m \not\equiv 3 \bmod 4$, see Examples 5.3 and 5.7). Unfortunately the methods used in the proofs of Propositions 1.7–1.9 break down for E_{4n+3,D_8}, and Section 7 discusses a few such aspects focusing attention on the case $n = 0$.

The spectral sequence methods in this paper are similar in spirit to those in [3] and [9]. In the latter reference, Feichtner and Ziegler describe the integral cohomology rings of *ordered* configuration spaces on spheres by means of a full analysis of the Serre spectral sequence (SSS) associated to the Fadell-Neuwirth fibration $\pi\colon F(S^k, n) \to S^k$ given by $\pi(x_1, \ldots, x_n) = x_n$ (a similar study is carried out in [10], but in the context of *ordered* orbit configuration spaces). One of the main achievements of the present paper is a successful calculation of cohomology groups of *unordered* configuration spaces (on real projective spaces), where no Fadell-Neuwirth fibrations are available—instead we rely on Lemma 1.6 and the CLSS[2] of the G-action on $V_{m+1,2}$. Also worth

[2] Our CLSS calculations can also be done in terms of the SSS of the fibration $V_{m+1,2} \xrightarrow{\theta} E_{m,G} \xrightarrow{p} BG$.

stressing is the fact that we succeed in computing cohomology groups with *integer* coefficients, whereas the Leray spectral sequence (and its Σ_k-invariant version) for the inclusion $F(X,k) \hookrightarrow X^k$ has proved to be effectively computable mainly when *field* coefficients are used ([11, 28]).

A major obstacle we have to confront (not present in [9]) comes from the fact that the spectral sequences we encounter often have non-simple systems of local coefficients. This is also the situation in [3], where the two-hyperplane case of Grünbaum's mass partition problem ([14]) is studied from the Fadell-Husseini index theory viewpoint [7]. Indeed, Blagojević and Ziegler deal with twisted coefficients in their main SSS, namely the one associated to the Borel fibration

$$S^m \times S^m \to ED_8 \times_{D_8} (S^m \times S^m) \xrightarrow{\overline{p}} BD_8 \tag{3}$$

where the D_8-action on $S^m \times S^m$ is the obvious extension of that in Definition 1.4. Now, the main goal in [3] is to describe the kernel of the map induced by $\overline{p}$ in integral cohomology—the so-called Fadell-Husseini ($\mathbb{Z}$-)index of D_8 acting on $S^m \times S^m$, $\mathrm{Index}_{D_8}(S^m \times S^m)$. Since D_8 acts freely on $V_{m+1,2}$, $\mathrm{Index}_{D_8}(S^m \times S^m)$ is contained in the kernel of the map induced in integral cohomology by the map $p\colon E_m \to BD_8$ in Proposition 1.9 (whether or not $m \equiv 3 \bmod 4$). In particular, the work in [3] can be used to identify explicit elements in $\mathrm{Ker}(p^*)$ and, as observed in Remark 1.10, our approach allows us to assess, for $m \not\equiv 3 \bmod 4$ (in Examples 5.3 and 5.7), how much of the actual kernel is still lacking description: [3] gives just a bit less than half the expected elements in $\mathrm{Ker}(p^*)$.

2 Preliminary cohomology facts

As shown in [1] (see also [15] for a straightforward approach), the mod 2 cohomology of D_8 is a polynomial ring on three generators $x, x_1, x_2 \in H^*(BD_8;\mathbb{F}_2)$, the first two of dimension 1, and the last one of dimension 2, subject to the single relation $x^2 = x \cdot x_1$. The classes x_i are the restrictions of the universal Stiefel-Whitney classes w_i ($i = 1,2$) under the map corresponding to the group inclusion $D_8 \subset \mathrm{O}(2)$ in Definition 1.4. On the other hand, the class x is not characterized by the relation $x^2 = x \cdot x_1$, but by the requirement that, for all m, x pulls back, under the map p_{m,D_8} in (2), to the map $u\colon B(\mathrm{P}^m, 2) \to \mathrm{P}^\infty$ classifying the obvios double cover $F(\mathrm{P}^m, 2) \to B(\mathrm{P}^m, 2)$—see [15, Proposition 3.5]. In particular:

Lemma 2.1. *For $i \geq 0$, $H^i(BD_8;\mathbb{F}_2) = \langle i+1 \rangle$.* $\square$

Corollary 2.2. *For any m,*

$$H^i(B(\mathrm{P}^m,2);\mathbb{F}_2) = \begin{cases} \langle i+1 \rangle, & 0 \leq i \leq m-1; \\ \langle 2m-i \rangle, & m \leq i \leq 2m-1; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. The assertion for $i \geq 2m$ follows from Lemma 1.6 and dimensional considerations. Poincaré duality implies that the assertion for $m \leq i \leq 2m-1$ follows from that for $0 \leq i \leq m-1$. Since $V_{m+1,2}$ is $(m-2)$-connected, the assertion for $0 \leq i \leq m-1$ follows from Lemma 2.1, using the fact (a consequence of [15, Proposition 3.6 and (3.8)]) that, in the mod 2 SSS for the fibration $V_{m+1,2} \xrightarrow{\theta} E_{m,D_8} \xrightarrow{p} BD_8$, the two indecomposable elements in $H^*(V_{m+1,2};\mathbb{F}_2)$ transgress to nontrivial elements. $\square$

Let $\mathbb{Z}_\alpha$ denote the $\mathbb{Z}[D_8]$-module whose underlying group is free on a generator α on which each of $\rho, \rho_1, \rho_2 \in D_8$ acts via multiplication by -1 (in particular, elements in $D_8 \cap \mathrm{SO}(2)$ act trivially). Corollaries 2.4 and 2.5 below are direct consequences of the following description, proved in [16] (see also [3, Theorem 4.5]), of the ring $H^*(BD_8)$ and of the $H^*(BD_8)$-module $H^*(BD_8;\mathbb{Z}_\alpha)$:

Theorem 2.3 (Handel [16])**.** *$H^*(BD_8)$ is generated by classes μ_2, ν_2, λ_3, and κ_4 subject to the relations $2\mu_2 = 2\nu_2 = 2\lambda_3 = 4\kappa_4 = 0$, $\nu_2^2 = \mu_2\nu_2$, and $\lambda_3^2 = \mu_2\kappa_4$. $H^*(BD_8;\mathbb{Z}_\alpha)$ is the free $H^*(BD_8)$-module on classes α_1 and α_2 subject to the relations $2\alpha_1 = 4\alpha_2 = 0$, $\lambda_3\alpha_1 = \mu_2\alpha_2$, and $\kappa_4\alpha_1 = \lambda_3\alpha_2$. Subscripts in the notation of these six generators indicate their cohomology dimensions.* $\square$

The notation a_2, b_2, c_3, and d_4 was used in [16] instead of the current μ_2, ν_2, λ_3, and κ_4. The change is made in order to avoid confusion with the generic notation d_i for differentials in the several spectral sequences considered in this paper.

Corollary 2.4. *For $a \geq 0$ and $0 \leq b \leq 3$,*

$$H^{4a+b}(BD_8) = \begin{cases} \mathbb{Z}, & (a,b) = (0,0); \\ \{2a\}, & b = 0 < a; \\ \langle 2a \rangle, & b = 1; \\ \langle 2a+2 \rangle, & b = 2; \\ \langle 2a+1 \rangle, & b = 3. \end{cases}$$

$\square$

Corollary 2.5. *For $a \geq 0$ and $0 \leq b \leq 3$,*

$$H^{4a+b}(BD_8;\mathbb{Z}_\alpha) = \begin{cases} \langle 2a\rangle, & b=0; \\ \langle 2a+1\rangle, & b=1; \\ \{2a\}, & b=2; \\ \langle 2a+2\rangle, & b=3. \end{cases} \qquad \square$$

We show that, up to a certain symmetry condition (exemplified in Table 1 at the end of Section 4), the groups explicitly described by Corollaries 2.4 and 2.5 delineate the additive structure of the graded group $H^*(B(\mathrm{P}^m,2))$. The corresponding situation for $H^*(F(\mathrm{P}^m,2))$ uses the following well-known analogues of Lemma 2.1 and Corollaries 2.2, 2.4 and 2.5:

Lemma 2.6. *For $i \geq 0$, $H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty;\mathbb{F}_2) = \langle i+1\rangle$.* $\square$

Lemma 2.7. *For any m,*

$$H^i(F(\mathrm{P}^m,2);\mathbb{F}_2) = \begin{cases} \langle i+1\rangle, & 0 \leq i \leq m-1; \\ \langle 2m-i\rangle, & m \leq i \leq 2m-1; \\ 0, & \textit{otherwise}. \end{cases} \qquad \square$$

Lemma 2.8. *For $i \geq 0$,*

$$H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty) \;=\; \begin{cases} \mathbb{Z}, & i=0; \\ \left\langle \frac{i}{2}+1\right\rangle, & i \textit{ even}, i>0; \\ \left\langle \frac{i-1}{2}\right\rangle, & \textit{otherwise}. \end{cases}$$

$$H^i(\mathrm{P}^\infty \times \mathrm{P}^\infty;\mathbb{Z}_\alpha) \;=\; \begin{cases} \left\langle \frac{i}{2}\right\rangle, & i \textit{ even}; \\ \left\langle \frac{i+1}{2}\right\rangle, & i \textit{ odd}. \end{cases}$$

Here $\mathbb{Z}_\alpha$ is regarded as a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-module via the restricted structure coming from the inclusion $\mathbb{Z}_2 \times \mathbb{Z}_2 \hookrightarrow D_8$. $\square$

Here are some brief comments on the proofs of Lemmas 2.6–2.8. Of course, the ring structure $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty;\mathbb{F}_2) = \mathbb{F}_2[x_1,y_1]$ is standard (as in Theorem 2.3, subscripts for the cohomology classes in this paragraph indicate dimension). On the other hand, it is easily shown (see for instance [17, Example 3E.5 on pages 306–307]) that $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty)$ is

the polynomial ring over the integers on three classes x_2, y_2, and z_3 subject to the four relations

$$(4) \qquad 2x_2 = 0, \ \ 2y_2 = 0, \ \ 2z_3 = 0, \text{ and } \ z_3^2 = x_2y_2(x_2 + y_2).$$

These two facts yield Lemma 2.6 and the first equality in Lemma 2.8. Lemma 2.7 can be proved with the argument given for Corollary 2.2—replacing D_8 by its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (1). Finally, both equalities in Lemma 2.8 can be obtained as immediate consequences of the Künneth exact sequence (for the second equality, note that $\mathbb{Z}_\alpha$ arises as the tensor square of the standard twisted coefficients for a single factor P^∞).

Remark 2.9. For future reference we recall (again from Hatcher's book) that the mod 2 reduction map $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty) \to H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{F}_2)$, a monomorphism in positive dimensions, is characterized by $x_2 \mapsto x_1^2$, $y_2 \mapsto y_1^2$, and $z_3 \mapsto x_1y_1(x_1 + y_1)$.

3 Orientability properties of some quotients of $V_{n,2}$

Proofs in this section will be postponed until all relevant results have been presented. Recall that all Stiefel manifolds $V_{n,2}$ are orientable (actually parallelizable, cf. [26]). Even if some of the elements of a given subgroup H of $\mathrm{O}(2)$ fail to act on $V_{n,2}$ in an orientation-preserving way, we could still use the possible orientability of the quotients $V_{n,2}/H$ as an indication of the extent to which H, as a whole, is compatible with the orientability of the several $V_{n,2}$. For example, while every element of $\mathrm{SO}(2)$ gives an orientation-preserving diffeomorphism on each $V_{n,2}$, it is well known that the Grassmannian $V_{n,2}/\mathrm{O}(2)$ of unoriented 2-planes in $\mathbb{R}^n$ is orientable if and only if n is even (see for instance [23, Example 47 on page 162]). We show that a similar—but *shifted*—result holds when $\mathrm{O}(2)$ is replaced by D_8.

Notation 3.1. For a subgroup H of $\mathrm{O}(2)$, we will use the shorthand $V_{n,H}$ to denote the quotient $V_{n,2}/H$. For instance $V_{m+1,G} = E_{m,G}$, the space in Notation 1.5.

Proposition 3.2. *For $n > 2$, V_{n,D_8} is orientable if and only if n is odd. Consequently, for $m > 1$, the top dimensional cohomology group of $B(\mathrm{P}^m, 2)$ is*

$$H^{2m-1}(B(\mathrm{P}^m, 2)) = \begin{cases} \mathbb{Z}, & \text{for even } m; \\ \mathbb{Z}_2, & \text{for odd } m. \end{cases}$$

Remark 3.3. Proposition 3.2 holds (with the same proof) if D_8 is replaced by its subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $B(\mathrm{P}^m, 2)$ is replaced by $F(\mathrm{P}^m, 2)$. It is interesting to compare both versions of Proposition 3.2 with the fact that, for $m > 1$, $B(\mathrm{P}^m, 2)$ is non-orientable, while $F(\mathrm{P}^m, 2)$ is orientable only for odd m ([18, Lemma 2.6]).

Example 3.4. The cases with $n = 2$ and $m = 1$ in Proposition 3.2 are special (compare to [18, Proposition 2.5]): Since the quotient of $V_{2,2} = S^1 \cup S^1$ by the action of $D_8 \cap \mathrm{SO}(2)$ is diffeomorphic to the disjoint union of two copies of $S^1/\mathbb{Z}_4$, we see that $V_{2,D_8} \cong S^1$.

If we take the same orientation for both circles in $V_{2,2} = S^1 \cup S^1$, it is clear that the automorphism $H^1(V_{2,2}) \to H^1(V_{2,2})$ induced by an element $r \in D_8$ is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if $r \in \mathrm{SO}(2)$, but by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ if $r \notin \mathrm{SO}(2)$. For larger values of n, the method of proof of Proposition 3.2 allows us to describe the action of D_8 on the integral cohomology ring of $V_{n,2}$. The answer is given in terms of the generators $\rho, \rho_1, \rho_2 \in D_8$ introduced in Definition 1.4.

Theorem 3.5. *The three automorphisms $\rho^*, \rho_1^*, \rho_2^* \colon H^q(V_{n,2}) \to H^q(V_{n,2})$ agree. For $n > 2$, this common morphism is the identity except when n is even and $q \in \{n-2, 2n-3\}$, in which case the common morphism is multiplication by -1.*

Theorem 3.5 should be read keeping in mind the well-known cohomology ring $H^*(V_{n,2})$. We recall its simple description after proving Proposition 3.2. For the time being it suffices to recall, for the purposes of Proposition 3.6 below, that $H^{n-1}(V_{n,2}) = \mathbb{Z}_2$ for odd n, $n \geq 3$.

We use our approach to Theorem 3.5 in order to describe the integral cohomology ring of the oriented Grassmannian $V_{n,\mathrm{SO}(2)}$ for odd n, $n \geq 3$. Although the result might be well known ($V_{n,\mathrm{SO}(2)}$ is a complex quadric of complex dimension $n-2$), we include the details (an easy step from the constructions in this section) since we have not been able to find an explicit reference in the literature.

Proposition 3.6. *Assume n is odd, $n = 2a + 1$ with $a \geq 1$. Let $\widetilde{z} \in H^2(V_{n,\mathrm{SO}(2)})$ stand for the Euler class of the smooth principal S^1-bundle*

$$S^1 \to V_{n,2} \to V_{n,\mathrm{SO}(2)} \tag{5}$$

There is a class $\widetilde{x} \in H^{n-1}(V_{n,\mathrm{SO}(2)})$ *mapping under the projection in* (5) *to the nontrivial element in* $H^{n-1}(V_{n,2})$*. Furthermore, as a ring*

$$H^*(V_{n,\mathrm{SO}(2)}) = \mathbb{Z}[\widetilde{x}, \widetilde{z}\,]/I_n$$

where I_n *is the ideal generated by*

$$\widetilde{x}^{\,2}, \quad \widetilde{x}\,\widetilde{z}^{\,a}, \quad \textit{and} \quad \widetilde{z}^{\,a} - 2\widetilde{x}. \tag{6}$$

It should be noted that the second generator of I_n is superfluous. We include it in the description since it will become clear, from the proof of Proposition 3.6, that the first two terms in (6) correspond to the two families of differentials in the SSS of the fibration classifying (5), while the last term corresponds to the family of nontrivial extensions in the resulting E_∞-term.

Remark 3.7. It is illuminating to compare Proposition 3.6 with H. F. Lai's computation of the cohomology ring $H^*(V_{n,\mathrm{SO}(2)})$ for even n, $n \geq 4$. According to [20, Theorem 2], $H^*(V_{2a,\mathrm{SO}(2)}) = \mathbb{Z}[\kappa, \widetilde{z}\,]/I_{2a}$ where I_{2a} is the ideal generated by

$$\kappa^2 - \varepsilon\kappa\widetilde{z}^{\,a-1} \quad \text{and} \quad \widetilde{z}^{\,a} - 2\kappa\widetilde{z}. \tag{7}$$

Here $\varepsilon = 0$ for a even, and $\varepsilon = 1$ for a odd, while the generator $\kappa \in H^{2a-2}(V_{2a,\mathrm{SO}(2)})$ is the Poincaré dual of the homology class represented by the canonical (realification) embedding $\mathbb{C}\mathrm{P}^{a-1} \hookrightarrow V_{2a,\mathrm{SO}(2)}$ (Lai also proves that $(-1)^{a-1}\kappa\widetilde{z}^{\,a-1}$ is the top dimensional cohomology class in $V_{2a,\mathrm{SO}(2)}$ corresponding to the canonical orientation of this manifold). The first fact to observe in Lai's description of $H^*(V_{2a,\mathrm{SO}(2)})$ is that the two dimensionally forced relations $\kappa\widetilde{z}^{\,a} = 0$ and $\widetilde{z}^{\,2a-1} = 0$ can be algebraically deduced from the relations implied by (7). A similar situation holds for $H^*(V_{2a+1,\mathrm{SO}(2)})$, where the first two relations in (6), as well as the corresponding algebraically implied relation $\widetilde{z}^{\,2a} = 0$, are forced by dimensional considerations. But it is more interesting to compare Lai's result with Proposition 3.6 through the canonical inclusions $\iota_n\colon V_{n,\mathrm{SO}(2)} \hookrightarrow V_{n+1,\mathrm{SO}(2)}$ $(n \geq 3)$. In fact, the relations given by the last element both in (6) and (7) readily give

$$\iota_{2a}^*(\widetilde{x}) = \kappa\widetilde{z} \quad \text{and} \quad \iota_{2a+1}^*(\kappa) = \widetilde{x} \tag{8}$$

for $a \geq 2$. Note that the second equality in (8) can be proved, for all $a \geq 1$, with the following alternative argument: From [20, Theorem 2],

$2\kappa - \widetilde{z}^a \in V_{2a+2,\mathrm{SO}(2)}$ is the Euler class of the canonical *normal* bundle of $V_{2a+2,\mathrm{SO}(2)}$ and, therefore, maps trivially under ι^*_{2a+1}. The second equality in (8) then follows from the relation implied by the last element in (6). Needless to say, the usual cohomology ring $H^*(B\mathrm{SO}(2))$ is recovered as the inverse limit of the maps ι^*_n (of course $B\mathrm{SO}(2) \simeq \mathbb{C}\mathrm{P}^\infty$).

Proof of Proposition 3.2 *from Theorem* 3.5. Since the action of every element in $D_8 \cap \mathrm{SO}(2)$ preserves orientation in $V_{n,2}$, and since two elements in $D_8 - \mathrm{SO}(2)$ must "differ" by an orientation-preserving element in D_8, the first assertion in Proposition 3.2 will follow once we argue that (say) ρ is orientation-preserving precisely when n is odd. But such a fact is given by Theorem 3.5 in view of the UCT. The second assertion in Proposition 3.2 then follows from Lemma 1.6, [17, Corollary 3.28], and the UCT (recall $\dim(V_{n,2}) = 2n-3$). □

We now start working toward the proof of Theorem 3.5, recalling in particular the cohomology ring $H^*(V_{n,2})$. Let $n > 2$ and think of $V_{n,2}$ as the sphere bundle of the tangent bundle of S^{n-1}. The (integral cohomology) SSS for the fibration $S^{n-2} \xrightarrow{\iota} V_{n,2} \xrightarrow{\pi} S^{n-1}$ (where $\pi(v_1, v_2) = v_1$ and $\iota(w) = (e_1, (0, w))$ with $e_1 = (1, 0, \ldots, 0)$) starts as

$$(9) \quad E_2^{p,q} = \begin{cases} \mathbb{Z}, & (p,q) \in \{(0,0), (n-1,0), (0,n-2), (n-1,n-2)\}; \\ 0, & \text{otherwise;} \end{cases}$$

and the only possibly nonzero differential is multiplication by the Euler characteristic of S^{n-1} (see for instance [21, pages 153–154]). At any rate, the only possibilities for a nonzero cohomology group $H^q(V_{n,2})$ are $\mathbb{Z}_2$ or $\mathbb{Z}$. In the former case, any automorphism must be the identity. So the real task is to determine the action of the three elements in Theorem 3.5 on a cohomology group $H^q(V_{n,2}) = \mathbb{Z}$.

Proof of Theorem 3.5. The fact that $\rho^* = \rho_1^* = \rho_2^*$ follows by observing that the product of any two of the elements ρ, ρ_1, and ρ_2 lies in the path connected group $\mathrm{SO}(2)$, and therefore determines an automorphism $V_{n,2} \to V_{n,2}$ which is homotopic to the identity.

The analysis of the second assertion of Theorem 3.5 depends on the parity of n.

Case with n even, $n > 2$. The SSS (9) collapses, giving that $H^*(V_{n,2})$ is an exterior algebra (over $\mathbb{Z}$) on a pair of generators x_{n-2} and x_{n-1} (indices denote dimensions). The spectral sequence also gives that x_{n-2}

maps under ι^* to the generator in S^{n-2}, whereas x_{n-1} is the image under π^* of the generator in S^{n-1}. Now, the (obviously) commutative diagram

$$\begin{tikzcd}
S^{n-2} \arrow[rr, "\text{antipodal map}"] \arrow[d, "\iota"'] & & S^{n-2} \arrow[d, "\iota"] \\
V_{n,2} \arrow[rr, "\rho_2"] \arrow[dr, "\pi"'] & & V_{n,2} \arrow[dl, "\pi"] \\
& S^{n-1} &
\end{tikzcd}$$

implies that ρ_2^* (and therefore ρ_1^* and ρ^*) is the identity on $H^{n-1}(V_{n,2})$, and that ρ_2^* (and therefore ρ_1^* and ρ^*) act by multiplication by -1 on $H^{n-2}(V_{n,2})$. The multiplicative structure then implies that the last assertion holds also on $H^{2n-3}(V_{n,2})$.

Case with n odd, $n > 2$. The description in (9) of the start of the SSS implies that the only nonzero cohomology groups of $V_{n,2}$ are $H^{n-1}(V_{n,2}) = \mathbb{Z}_2$ and $H^i(V_{n,2}) = \mathbb{Z}$ for $i = 0, 2n-3$. Thus, we only need to make sure that

(10) $$\rho^* \colon H^{2n-3}(V_{n,2}) \to H^{2n-3}(V_{n,2}) \text{ is the identity morphism.}$$

Choose generators $x \in H^{n-1}(V_{n,2})$, $y \in H^{2n-3}(V_{n,2})$, and $z \in H^2(\mathbb{C}\mathrm{P}^\infty)$, and let $V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty$ classify the circle fibration (5). Thus, the E_2-term of the SSS for the fibration

(11) $$V_{n,2} \to V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty$$

takes the simple form

$$\begin{array}{c|ccccccccccccc}
y & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
x & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots \\
 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
 & 1 & z & z^2 & z^3 & \cdots & z^{a-1} & z^a & z^{a+1} & \cdots & z^{n-2} & z^{n-1} & z^n & \cdots
\end{array}$$

where $n = 2a + 1$, and a bullet represents a copy of $\mathbb{Z}_2$. The proof of Proposition 3.6 below gives two rounds of differentials, both originating

on the top horizontal line; the element $2y$ is a cycle in the first round of differentials, but determines the second round of differentials by

$$d_{2n-2}(2y) = z^{n-1}. \tag{12}$$

The key ingredient comes from the observation that ρ and the involution $\tau\colon V_{n,\mathrm{SO}(2)} \to V_{n,\mathrm{SO}(2)}$ that reverses orientation of an oriented 2-plane fit into the pull-back diagram

$$\begin{tikzcd} V_{n,2} \arrow[r, "\rho"] \arrow[d] & V_{n,2} \arrow[d] \\ V_{n,\mathrm{SO}(2)} \arrow[r, "\tau"] \arrow[d] & V_{n,\mathrm{SO}(2)} \arrow[d] \\ \mathbb{C}\mathrm{P}^\infty \arrow[r, "c"] & \mathbb{C}\mathrm{P}^\infty \end{tikzcd} \tag{13}$$

where c stands for conjugation. [Indeed, thinking of $V_{n,\mathrm{SO}(2)} \to \mathbb{C}\mathrm{P}^\infty$ as an inclusion, τ is the restriction of c, and ρ becomes the equivalence induced on (selected) fibers.] Of course $c^*(z) = -z$ in $H^2(\mathbb{C}\mathrm{P}^\infty)$, so that

$$c^*(z^{n-1}) = z^{n-1} \tag{14}$$

(recall n is odd). Thus, in terms of the map of spectral sequences determined by (13), conditions (12) and (14) force the relation $\rho^*(2y) = 2y$. This gives (10). □

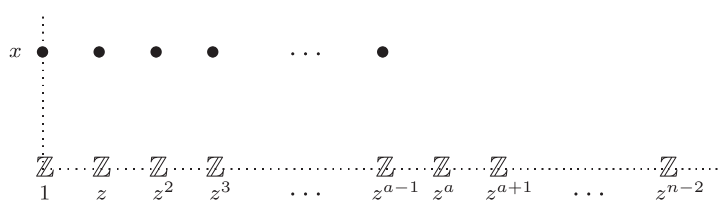
The proof of (10) we just gave (for odd n) can be simplified by working over the rationals (see Remark 3.8 in the next paragraph). We have chosen the spectral sequence analysis of (11) since it leads us to Proposition 3.6.

Remark 3.8. It is well known that whenever a finite group H acts freely on a space X, with $Y = X/H$, the rational cohomology of Y maps isomorphically onto the H-invariant elements in the rational cohomology of X (see for instance [17, Proposition 3G.1]). We apply this fact to the 8-fold covering projection $\theta\colon V_{n,2} \to V_{n,D_8}$. Since the only nontrivial groups $H^q(V_{n,2};\mathbb{Q})$ are $\mathbb{Q}$ for $q = 0, 2n-3$ (this is where we use that n is odd), we get that the rational cohomology of V_{n,D_8} is $\mathbb{Q}$ in dimension 0, vanishes in positive dimensions below $2n-3$, and is either $\mathbb{Q}$ or 0 in

the top dimension $2n-3$. But V_{n,D_8} is a manifold of odd dimension, so its Euler characteristic is zero; this forces the top rational cohomology to be $\mathbb{Q}$. Thus, every element in D_8 acts as the identity on the top rational (and therefore integral) cohomology group of $V_{n,2}$. This gives in particular (10), the real content of Theorem 3.5 for an odd n.

As in the notation introduced right after (10), let $z \in H^2(\mathbb{C}\mathrm{P}^\infty)$ be a generator so that the element $\widetilde{z} \in H^2(V_{n,\mathrm{SO}(2)})$ in Proposition 3.6 is the image of z under the projection map in (11).

Proof of Proposition 3.6. The E_2-term of the SSS for (11) has been indicated in the proof of Theorem 3.5. In that picture, the horizontal x-line consists of permanent cycles; indeed, there is no nontrivial target in a $\mathbb{Z}$ group for a differential originating at a $\mathbb{Z}_2$ group. Since $\dim(V_{n,\mathrm{SO}(2)}) = 2n-4$, the term xz^a must be killed by a differential, and the only way this can happen is by means of $d_{n-1}(y) = xz^a$. By multiplicativity, this settles a whole family of differentials killing off the elements xz^i with $i \geq a$. Note that this still leaves groups $2 \cdot \mathbb{Z}$ in the y-line (rather, the $2y$-line). Just as before, dimensionality forces the differential (12), and multiplicativity determines a corresponding family of differentials. What remains in the SSS after these two rounds of differentials—depicted below—consists of permanent cycles, so the spectral sequence collapses from this point on.



Finally, we note that all possible extensions are nontrivial. Indeed, orientability of $V_{n,\mathrm{SO}(2)}$ gives $H^{2n-4}(V_{n,\mathrm{SO}(2)}) = \mathbb{Z}$, which implies a nontrivial extension involving xz^{a-1} and z^{n-2}. Since multiplication by z is monic in total dimensions less that $2n-4$ of the E_∞-term, the 5-Lemma (applied recursively) shows that the same assertion is true in $H^*(V_{n,\mathrm{SO}(2)})$. This forces the corresponding nontrivial extensions in degrees lower than $2n-4$: an element of order 2 in low dimensions would produce, after multiplication by z, a corresponding element of order 2 in the top dimension. The proposition follows. $\square$

Lai's description of the ring $H^*(V_{2a,\mathrm{SO}(2)})$ given in Remark 3.7 can be used to understand the full patter of differentials and extensions in the SSS of (11) for $n = 2a$. Due to space limitations, details are not given here—but they are discussed in Remark 3.10 of the preliminary version [13] of this paper.

We close this section with an argument that explains, in a geometric way, the switch in parity of n when comparing the orientability properties of $V_{n,\mathrm{O}(2)}$ to those of V_{n,D_8}. Let π stand for the projection map in the smooth fiber bundle (5). The tangent bundle $T_{n,2}$ to $V_{n.2}$ decomposes as the Whitney sum

$$T_{n,2} \cong \pi^*(T_{n,\mathrm{SO}(2)}) \oplus \lambda$$

where $T_{n,\mathrm{SO}(2)}$ is the tangent bundle to $V_{n,\mathrm{SO}(2)}$, and λ is the 1–dimensional bundle of tangents to the fibers—a trivial bundle since we have the nowhere vanishing vector field obtained by differentiating the free action of S^1 on $V_{n.2}$. Note that $\rho\colon V_{n,2} \to V_{n,2}$ reverses orientation on all fibers and so reverses a given orientation of λ. Hence, ρ *preserves* a chosen orientation of $T_{n,2}$ precisely when the involution τ in (13) *reverses* a chosen orientation of $T_{n,\mathrm{SO}(2)}$. But, as explained in the proof of Proposition 3.2, V_{n,D_8} is orientable precisely when ρ is orientation-preserving. Likewise, $V_{n,\mathrm{O}(2)}$ is orientable precisely when τ is orientation-preserving.

4 Torsion linking form and Theorems 1.1 and 1.2

In this short section we outline an argument, based on the classical torsion linking form, that allows us to compute the cohomology groups described by Theorems 1.1 and 1.2 in all but three critical dimensions. The totality of dimensions (together with the proofs of Propositions 1.7–1.9) is considered in the next three sections—the first two of which represent the bulk of spectral sequence computations in this paper.

For a space X let $TH_i(X;A)$ (respectively, $TH^i(X;A)$) denote the torsion subgroup of the i^{th} homology (respectively, cohomology) group of X with (possibly twisted) coefficients A. As usual, omission of A from the notation indicates that a simple system of $\mathbb{Z}$-coefficients is used. We are interested in the twisted coefficients $\widetilde{\mathbb{Z}}$ arising from the orientation

character of a closed m-manifold $X = M$ for, in such a case, there are non-singular pairings

$$TH^i(M) \times TH^j(M; \widetilde{\mathbb{Z}}) \to \mathbb{Q}/\mathbb{Z} \tag{15}$$

(for $i+j = m+1$), the so-called torsion linking forms, constructed from the UCT and Poincaré duality. Although (15) seems to be best known for an orientable M (see for instance [27, pages 16–17 and 58–59]), the construction works just as well in a non-orientable setting. We briefly recall the details (in cohomological terms) for completeness.

Start by observing that for a finitely generated abelian group $H = F \oplus T$ with F free abelian and T a finite group, the group $\mathrm{Ext}^1(H, \mathbb{Z}) \cong \mathrm{Ext}^1(T, \mathbb{Z})$ is canonically isomorphic to $\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$, the Pontryagin dual of T (verify this by using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, and noting that $\mathbb{Q}$ is injective while $\mathrm{Hom}(T, \mathbb{Q}) = 0$). In particular, the canonical isomorphism $TH^i(M) \cong \mathrm{Ext}^1(TH_{i-1}(M), \mathbb{Z})$ coming from the UCT yields a non-singular pairing $TH^i(M) \times TH_{i-1}(M) \to \mathbb{Q}/\mathbb{Z}$. The form in (15) then follows by using Poincaré duality (in its not necessarily orientable version, see [17, Theorem 3H.6] or [24, Theorem 4.51]). As explained by Barden in [2, Section 0.7] (in the orientable case), the resulting pairing can be interpreted geometrically as the classical torsion linking number ([19, 25, 29]).

Recall the group G and orbit space E_m in Notation 1.5. We next indicate how the isomorphisms

$$TH^i(M) \cong TH^j(M; \widetilde{\mathbb{Z}}), \quad i + j = 2m, \tag{16}$$

coming from (15) for $M = E_m$ can be used for computing most of the integral cohomology groups of $F(\mathrm{P}^m, 2)$ and $B(\mathrm{P}^m, 2)$.

Since $V_{m+1,2}$ is $(m-2)$-connected[3], the map in (2) is $(m-1)$-connected. Therefore it induces an isomorphism (respectively, monomorphism) in cohomology with any—possibly twisted, in view of [30, Theorem 6.4.3*] —coefficients in dimensions $i \leq m-2$ (respectively, $i = m-1$). Together with Corollary 2.4 and Lemmas 1.6 and 2.8, this leads to the explicit description of the groups in Theorems 1.1 and 1.2 in dimensions at most $m-2$. The corresponding groups in dimensions at least $m+2$ can then be obtained from the isomorphisms (16) and the full description in

[3]Low-dimensional cases with $m \leq 3$ are given special attention in Example 5.1, Remark 5.4, and (32) in the following sections.

Section 2 of the twisted and untwisted cohomology groups of BG. Note that the last step requires knowing that, when E_m is non-orientable (as determined in Proposition 3.2 and Remark 3.3), the twisted coefficients $\widetilde{\mathbb{Z}}$ agree with those $\mathbb{Z}_\alpha$ used in Theorem 2.3. But such a requirement is a direct consequence of Theorem 3.5. Since the torsion-free subgroups of $H^*(E_m)$ are easily identifiable from a quick glance at the E_2-term of the CLSS for the G-action on $V_{m+1,2}$, only the torsion subgroups in Theorems 1.1 and 1.2 in dimensions

$$m-1,\ m,\ \text{and}\ \ m+1 \tag{17}$$

are lacking description in this argument.

A deeper analysis of the CLSS of the G-action on $V_{m+1,2}$ (worked out in Sections 5 and 6 for $G = D_8$, and discussed briefly in Section 8 for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$) will give us (among other things) a detailed description of the three missing cases in (17) *except* for the $(m+1)$-dimensional group when $G = D_8$ and $m \equiv 3 \bmod 4$. Note that this apparently singular case cannot be handled directly with the torsion linking form argument in the previous paragraph because the connectivity of $V_{m+1,2}$ only gives the injectivity, but not the surjectivity, of the first map in the composite

$$H^{m-1}(BD_8;\mathbb{Z}_\alpha) \xrightarrow{p^*} H^{m-1}(B(\mathrm{P}^m,2);\mathbb{Z}_\alpha) \cong H^{m+1}(B(\mathrm{P}^m,2)). \tag{18}$$

To overcome the problem, in Section 6 we perform a direct calculation in the first two pages of the Bockstein spectral sequence (BSS) of $B(\mathrm{P}^{4a+3},2)$ to prove that (18) is indeed an isomorphism for $m \equiv 3 \bmod 4$—therefore completing the proof of Theorems 1.1 and 1.2.

$*=$	2	3	4	5	6	7	8	9	10	11	12	13	14
$H^*(E_{2,D_8})$	$\langle 2\rangle$												
$H^*(E_{4,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$								
$H^*(E_{6,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 2\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$				
$H^*(E_{8,D_8})$	$\langle 2\rangle$	$\langle 1\rangle$	$\{2\}$	$\langle 2\rangle$	$\langle 4\rangle$	$\langle 3\rangle$	$\{4\}$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\{2\}$	$\langle 1\rangle$	$\langle 2\rangle$

Table 1: $H^*(E_{m,D_8}) \cong H^*(B(\mathrm{P}^m,2))$ for $m = 2,4,6$, and 8

The isomorphisms in (16) yield a (twisted, in the non-orientable case) symmetry for the torsion groups of $H^*(E_m)$. This is illustrated (for

$G = D_8$ and in the orientable case) in Table 1 following the conventions set in the very first paragraph of the paper.

5 Case of $B(\mathrm{P}^m, 2)$ for $m \not\equiv 3 \bmod 4$

This section and the next one contain a careful study of the CLSS of the D_8-action on $V_{m+1,2}$ described in Definition 1.4; the corresponding (much simpler) analysis for the restricted $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action is outlined in Section 8. The CLSS approach will yield, in addition, direct proofs of Propositions 1.7–1.9. The reader is assumed to be familiar with the properties of the CLSS of a regular covering space, complete details of which first appeared in [4].

We start with the less involved situation of an even m and, as a warm-up, we consider first the case $m = 2$.

Example 5.1. Lemmas 1.6 and 2.1, Corollary 2.4, and Theorem 3.5 imply that, in total dimensions at most $\dim(V_{3,D_8}) = 3$, the (integral cohomology) CLSS for the D_8-action on $V_{3,2}$ starts as

$$\begin{array}{c|cccc}
3 & \mathbb{Z} & & & \\
2 & \langle 1\rangle & \langle 2\rangle & & \\
1 & & & & \\
 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle \\
\hline
 & & 1 & 2 & 3
\end{array}$$

The only possible nontrivial differential in this range is $d_3^{0,2}\colon E_2^{0,2} \to E_2^{3,0}$, which must be an isomorphism in view of the second assertion in Proposition 3.2. This yields the P^2-case in Theorem 1.2 and Propositions 1.9 and 1.7 (with $G = D_8$ in the latter one). As indicated in Table 1, the symmetry isomorphisms are invisible in the current situation. It is worth noticing that the d_3-differential originating at node $(1,2)$ must be injective. This observation will be the basis in our argument for the general situation, where 2-rank considerations will be the catalyst. Here and in what follows, by the 2-*rank* (or simply *rank*) of a finite abelian 2-group H we mean the rank ($\mathbb{F}_2$-dimension) of $H \otimes \mathbb{F}_2$.

Proof of Proposition 1.7 *for* $G = D_8$*, and of Proposition* 1.9*, both for even* $m \geq 4$*.* The assertion in Proposition 1.7 for

- $i \geq 2m$ follows from Lemma 1.6 and the fact that $\dim(V_{m+1,2}) = 2m-1$, and for
- $i = 2m-1$ follows from the fact that $H^{2m-1}(BD_8)$ is a torsion group (Corollary 2.4) while $H^{2m-1}(B(\mathrm{P}^m,2)) = \mathbb{Z}$ (Proposition 3.2).

We work with the (integral cohomology) CLSS for the D_8-action on $V_{m+1,2}$ in order to prove Proposition 1.9 and the assertions in Proposition 1.7 for $i < 2m-1$.

In view of Theorem 3.5, the spectral sequence has a simple system of coefficients and, from the description of $H^*(V_{m+1,2})$ in the proof of Theorem 3.5, it is concentrated in the three horizontal lines with $q = 0, m, 2m-1$. We can focus on the lines with $q = 0, m$ in view of the range under current consideration. At the start of the CLSS there is a copy of

- $H^*(BD_8)$ (described by Corollary 2.4) at the line with $q = 0$;
- $H^*(BD_8, \mathbb{F}_2)$ (described by Lemma 2.1) at the line with $q = m$.

Note that the assertion in Proposition 1.7 for $i < m$ is an obvious consequence of the above description of the E_2-term of the CLSS. The case $i = m$ will follow once we show that the "first" potentially nontrivial differential $d_{m+1}^{0,m}\colon E_2^{0,m} \to E_2^{m+1,0}$ is injective. More generally, we show in the paragraph following (22) below that all differentials

(19) $\quad d_{m+1}^{m-\ell-1,m}\colon E_2^{m-\ell-1,m} \to E_2^{2m-\ell,0}$ with $0 < \ell < m$ are injective.

From this, the assertion in Proposition 1.7 for $m < i < 2m-1$ follows at once.

The information we need about differentials is forced by the "size" of their domains and codomains. For instance, since $H^{2m-1}(B(\mathrm{P}^m,2))$ is torsion-free, all of $E_2^{2m-1,0} = H^{2m-1}(BD_8) = \langle m-1\rangle$ must be killed by differentials. But the only possibly nontrivial differential landing in $E_2^{2m-1,0}$ is the one in (19) with $\ell = 1$. The resulting surjective $d_{m+1}^{m-2,m}$ map must be an isomorphism since its domain, $E_2^{m-2,m} = H^{m-2}(BD_8;\mathbb{F}_2) = \langle m-1\rangle$, is isomorphic to its codomain.

The extra input we need in order to deal with the rest of the differentials in (19) comes from the short exact sequences

(20) $\quad 0 \to \operatorname{Coker}(2_i) \to H^i(B(\mathrm{P}^m,2);\mathbb{F}_2) \to \operatorname{Ker}(2_{i+1}) \to 0$

obtained from the Bockstein long exact sequence

$$\cdots \leftarrow H^i(B(\mathrm{P}^m,2);\mathbb{F}_2) \overset{\pi_i}{\leftarrow} H^i(B(\mathrm{P}^m,2))$$
$$\overset{2_i}{\leftarrow} H^i(B(\mathrm{P}^m,2)) \overset{\partial_i}{\leftarrow} H^{i-1}(B(\mathrm{P}^m,2);\mathbb{F}_2) \leftarrow \cdots .$$

From the E_2-term of the spectral sequence we easily see that

$$H^1(B(\mathrm{P}^m,2)) = 0$$

and that

$$H^i(B(\mathrm{P}^m,2))$$

is a finite 2-torsion group for $1 < i < 2m-1$; let r_i denote its 2-rank. Then $\mathrm{Ker}(2_i) \cong \mathrm{Coker}(2_i) \cong \langle r_i \rangle$, so that (20), Corollary 2.2, and an easy induction (grounded by the fact that $\mathrm{Ker}(2_{2m-1}) = 0$, in view of the second assertion in Proposition 3.2) yield

$$r_{2m-\ell} = \begin{cases} a+1, & \ell = 2a; \\ a, & \ell = 2a+1; \end{cases} \tag{21}$$

for $2 \leq \ell \leq m-1$. Under these conditions, the ℓ-th differential in (19) takes the form

$$\langle m-\ell \rangle = H^{m-\ell-1}(BD_8;\mathbb{F}_2) \to H^{2m-\ell}(BD_8) \tag{22}$$

where

$$H^{2m-\ell}(BD_8) = \begin{cases} \left\{ m - \frac{\ell}{2} \right\}, & \ell \equiv 0 \bmod 4; \\ \left\langle m - \frac{\ell-2}{2} \right\rangle, & \ell \equiv 2 \bmod 4; \\ \left\langle m - \frac{\ell+1}{2} \right\rangle, & \text{otherwise.} \end{cases}$$

But the cokernel of this map, which is a subgroup of $H^{2m-\ell}(B(\mathrm{P}^m,2))$, must have 2-rank at most $r_{2m-\ell}$. An easy counting argument (using the right exactness of the tensor product) shows that this is possible only with an injective differential (22) which, in the case of $\ell \equiv 0 \bmod 4$, yields an injective map even after tensoring[4] with $\mathbb{Z}_2$.

Note that, in total dimensions at most $2m-2$, the E_{m+2}-term of the spectral sequence is concentrated on the base line ($q = 0$). Thus, for $2 \leq \ell \leq m-1$, $H^{2m-\ell}(B(\mathrm{P}^m,2))$ is the cokernel of the differential (22)—which yields the surjectivity asserted in Proposition 1.7 in the

[4]This amounts to the fact that twice the generator of the $\mathbb{Z}_4$-summand in (22) is not in the image of (22)—compare to the proof of Proposition 5.2.

range $m < i < 2m-1$. Furthermore the kernel of $p^*\colon H^{2m-\ell}(BD_8) \to H^{2m-\ell}(B(\mathrm{P}^m,2))$ is the elementary abelian 2-group specified on the left hand side of (22). In fact, the observation in the second half of the final assertion in the previous paragraph proves Proposition 1.9. $\square$

As indicated in the last paragraph of the previous proof, for $2 \leq \ell \leq m-1$ the CLSS analysis identifies the group $H^{2m-\ell}(B(\mathrm{P}^m,2))$ as the cokernel of (22). Thus, the following algebraic calculation of these groups not only gives us an alternative approach to that using the non-singularity of the torsion linking form, but it also allows us to recover (for m even and $G = D_8$) the three missing cases in (17)—therefore completing the proof of the P^{even}-case of Theorem 1.2.

Proposition 5.2. *For $2 \leq \ell \leq m-1$, the cokernel of the differential (22) is isomorphic to*

$$H^{2m-\ell}(B(\mathrm{P}^m,2)) = \begin{cases} \left\{\frac{\ell}{2}\right\}, & \ell \equiv 0 \bmod 4; \\ \left\langle\frac{\ell}{2}+1\right\rangle, & \ell \equiv 2 \bmod 4; \\ \left\langle\frac{\ell-1}{2}\right\rangle, & \textit{otherwise.} \end{cases}$$

Proof. Cases with $\ell \not\equiv 0 \bmod 4$ follow from a simple count, so we only offer an argument for $\ell \equiv 0 \bmod 4$. Consider the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \langle m-\ell\rangle \arrow[r] \arrow[d, equal] & \left\{m-\frac{\ell}{2}\right\} \arrow[r] & H^{2m-\ell}(B(\mathrm{P}^m,2)) \arrow[r] & 0 \\ 0 \arrow[r] & \langle m-\ell\rangle \arrow[r] & \left\langle m-\frac{\ell}{2}+1\right\rangle \arrow[r] \arrow[u, hook] & \left\langle\frac{\ell}{2}+1\right\rangle \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

where the top horizontal monomorphism is (22), and where the middle group on the bottom is included in the top one as the elements annihilated by multiplication by 2. The lower right group is $\langle\frac{\ell}{2}+1\rangle$ by a simple counting. The snake lemma shows that the right-hand-side vertical map is injective with cokernel $\mathbb{Z}_2$; the resulting extension is nontrivial in view of (21). $\square$

Example 5.3. For m even, [3, Theorem 1.4 (D)] identifies three explicit elements in the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m,2))$: one for each of $i = m+2$, $i = m+3$, and $i = m+4$. In particular, this produces at most four basis elements in the ideal $\mathrm{Ker}(p^*)$ in dimensions at most $m+4$.

However we have just seen that, for $m+1 \leq i \leq 2m-1$, the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m,2))$ is an $\mathbb{F}_2$-vector space of dimension $i-m$. This means that through dimensions at most $m+4$ (and with $m>4$) there are at least six more basis elements remaining to be identified in $\mathrm{Ker}(p^*)$.

We next turn to the case when m is odd (a hypothesis in force throughout the rest of the section) assuming, from Lemma 5.5 on, that $m \equiv 1 \bmod 4$.

Remark 5.4. Since the P^1-case in Proposition 1.9 and Theorems 1.2 and 1.8 is elementary (in view of Remark 1.3 and Corollary 2.4), we will implicitly assume $m \neq 1$.

The CLSS of the D_8-action on $V_{m+1,2}$ now has a few extra complications that turn the analysis of differentials into a harder task. To begin with, we find a twisted system of local coefficients (Theorem 3.5). As a $\mathbb{Z}[D_8]$-module, $H^q(V_{m+1,2})$ is:

- $\mathbb{Z}$ for $q=0,m$;
- $\mathbb{Z}_\alpha$ for $q=m-1,2m-1$;
- the zero module otherwise.

Thus, in total dimensions at most $2m-2$ the CLSS is concentrated on the three horizontal lines with $q=0,m-1,m$. [This is in fact the case in total dimensions at most $2m-1$, since $H^0(BD_8;\mathbb{Z}_\alpha)=0$; this observation is not relevant for the actual group $H^{2m-1}(B(\mathrm{P}^m,2)) = \mathbb{Z}_2$—given in the second assertion in Proposition 3.2—, but it will be relevant for the claimed surjectivity of the map $p^*\colon H^{2m-1}(BD_8) \to H^{2m-1}(B(\mathrm{P}^m,2))$.] In more detail, at the start of the CLSS we have a copy of $H^*(BD_8)$ at $q=0,m$, and a copy of $H^*(BD_8;\mathbb{Z}_\alpha)$ at $q=m-1$. It is the extra horizontal line at $q=m-1$ (not present for an even m) that leads to potential d_2-differentials—from the $(q=m)$-line to the $(q=m-1)$-line. Sorting these differentials out is the main difficulty (which we have been able to overcome only for $m \equiv 1 \bmod 4$). Throughout the remainder of the section we work in terms of this spectral sequence, making free use of the description of its E_2-term coming from Corollaries 2.4 and 2.5, as well as of its $H^*(BD_8)$-module structure. Note that the latter property implies that much of the global structure of the spectral sequence is dictated by differentials on the three elements

- $x_m \in E_2^{0,m} = H^0(BD_8; H^m(V_{m+1,2})) = H^0(BD_8;\mathbb{Z}) = \mathbb{Z}$;
- $\alpha_1 \in E_2^{1,m-1} = H^1(BD_8; H^{m-1}(V_{m+1,2})) = H^1(BD_8;\mathbb{Z}_\alpha) = \mathbb{Z}_2$;
- $\alpha_2 \in E_2^{2,m-1} = H^2(BD_8; H^{m-1}(V_{m+1,2})) = H^2(BD_8;\mathbb{Z}_\alpha) = \mathbb{Z}_4$;

each of which is a generator of the indicated group (notation is inspired by that in Theorem 2.3 and in the proof of Theorem 3.5—for even n).

Lemma 5.5. *For* $m \equiv 1 \bmod 4$ *and* $m \geq 5$, *the nontrivial* d_2-*differentials are given by* $d_2^{4i,m}(\kappa_4^i x_m) = 2\kappa_4^i\alpha_2$ *for* $i \geq 0$.

Proof. The only potentially nontrivial d_2-differentials originate at the $(q = m)$-line and, in view of the module structure, all we need to show is that

$$d_2\colon E_2^{0,m} \to E_2^{2,m-1} \text{ has } d_2(x_m) = 2\alpha_2 \tag{23}$$

(here and in what follows we omit superscripts of differentials).

Let $m = 4a + 1$. Since $H^{2m-1}(B(\mathrm{P}^m, 2)) = \langle 1\rangle$, most of the elements in $E_2^{2m-1,0} = \langle 4a\rangle$ must be wiped out by differentials. The only differentials landing in a $E_r^{2m-1,0}$ (that originate at a nonzero group) are

$$d_m\colon E_m^{m-1,m-1} \to E_m^{2m-1,0} \quad\text{and}\quad d_{m+1}\colon E_{m+1}^{m-2,m} \to E_{m+1}^{2m-1,0}. \tag{24}$$

But $E_2^{m-1,m-1} = \langle 2a\rangle$ and $E_2^{m-2,m} = \langle 2a-1\rangle$, so that rank considerations imply

$$E_2^{m-2,m} = E_{m+1}^{m-2,m}, \tag{25}$$

with the two differentials in (24) injective. In particular we get that

$$H^{2m-1}(B(\mathrm{P}^m, 2)) = \langle 1\rangle \text{ comes from } E_\infty^{2m-1,0} = \langle 1\rangle. \tag{26}$$

Furthermore, (25) and the $H^*(BD_8)$-module structure in the spectral sequence imply that the differential in (23) cannot be surjective.

It remains to show that the differential in (23) is nonzero. We shall obtain a contradiction by assuming that $d_2(x_m) = 0$, so that every element in the $(q = m)$-line is a d_2-cycle. Since $H^{2m}(B(\mathrm{P}^m, 2)) = 0$, all of $E_2^{2m,0} = \langle 4a + 2\rangle$ must be wiped out by differentials, and under the current hypothesis the only possible such differentials would be

$$d_m\colon E_m^{m,m-1} = E_2^{m,m-1} = \langle 2a+1\rangle \to E_m^{2m,0} = E_2^{2m,0}$$

and

$$d_{m+1}\colon E_{m+1}^{m-1,m} = E_2^{m-1,m} = \langle 2a\rangle \oplus \mathbb{Z}_4 \to E_{m+1}^{2m,0}$$

—indeed, $E_2^{0,2m-1} = H^0(BD_8;\mathbb{Z}_\alpha) = 0$. Thus, the former differential would have to be injective while the latter one would have to be surjective with a $\mathbb{Z}_2$ kernel. But there are no further differentials that could kill the resulting $E_{m+2}^{m-1,m} = \langle 1\rangle$, in contradiction to (26). □

Remark 5.6. In the preceding proof we made crucial use of the $H^*(BD_8)$-module structure in the spectral sequence in order to handle d_2-differentials. We show next that, just as in the proof of Proposition 1.7 for $G = D_8$, many of the properties of all higher differentials in the case $m \equiv 1 \bmod 4$ follow from the "size" of the resulting E_3-term.

Proof of Theorem 1.8 *for* $G = D_8$*, and of Proposition* 1.9*, both for* $m \equiv 1 \bmod 4$. The d_2 differentials in Lemma 5.5 replace, by a $\mathbb{Z}_2$-group, every instance of a $\mathbb{Z}_4$-group in the $(q = m-1)$ and $(q = m)$-lines of the E_2-term. This describes the E_3-term, the starting stage of the CLSS in the following considerations (note that the E_3-term agrees with the E_m-term). With this information the idea of the proof is formally the same as that in the case of an even m, namely: a little input from the Bockstein long exact sequence for $B(\mathrm{P}^m, 2)$ forces the injectivity of all relevant higher differentials (we give the explicit details for the reader's benefit).

Let $m = 4a + 1$ (recall we are assuming $a \geq 1$). The crux of the matter is showing that the differentials

$$d_m\colon E_3^{m-\ell,m-1} \to E_3^{2m-\ell,0} \text{ with } \ell = 0,1,2,\ldots,m \tag{27}$$

and

$$d_{m+1}\colon E_3^{m-\ell-1,m} \to E_{m+1}^{2m-\ell,0} \text{ with } \ell = 0,1,2,\ldots,m-1 \tag{28}$$

are injective and never hit twice the generator of a $\mathbb{Z}_4$-group. This assertion has already been shown for $\ell = 1$ in the paragraph containing (24). Likewise, the assertion for $\ell = 0$ follows from (26) with the same counting argument as the one used in the final paragraph of the proof of Lemma 5.5. Furthermore the case $\ell = m$ in (27) is obvious since $E_3^{0,m-1} = H^0(BD_8;\mathbb{Z}_\alpha) = 0$. However, since $E_3^{0,m} = H^0(BD_8) = \mathbb{Z}$ and $E_3^{m+1,0} = H^{m+1}(BD_8) = \langle 2a+2\rangle$, the injectivity assertion needs to be suitably interpreted for $\ell = m-1$ in (28); indeed, we will prove that

$$d_{m+1}\colon E_3^{0,m} \to E_{m+1}^{m+1,0} \tag{29}$$

yields an injective map *after* tensoring with $\mathbb{Z}_2$.

From the E_3-term of the spectral sequence we easily see that

$$H^m(B(\mathrm{P}^m, 2))$$

is the direct sum of a copy of $\mathbb{Z}$ and a finite 2-torsion group, while $H^i(B(\mathrm{P}^m, 2))$ is a finite 2-torsion group for $i \neq 0, m$. We consider the analogue of (20), the short exact sequences

$$0 \to \mathrm{Coker}(2_i) \to H^i(B(\mathrm{P}^m, 2); \mathbb{F}_2) \to \mathrm{Ker}(2_{i+1}) \to 0, \tag{30}$$

working here and below in the range $m + 1 \leq i \leq 2m - 2$. Let r_i denote the 2-rank of (the torsion subgroup of) $H^i(B(\mathrm{P}^m, 2))$, so that $\mathrm{Ker}(2_i) \cong \mathrm{Coker}(2_i) \cong \langle r_i \rangle$. Then Corollary 2.2, (30), and an easy induction (grounded by the fact that $\mathrm{Ker}(2_{2m-1}) = \langle 1 \rangle$, which in turn comes from the second assertion in Proposition 3.2) yield that

$$r_{2m-\ell} \text{ is the integral part of } \tfrac{\ell+1}{2} \text{ for } 2 \leq \ell \leq m - 1. \tag{31}$$

Now, in the range of (31), Lemma 5.5 and Corollaries 2.4 and 2.5 give

$$E_3^{m-\ell,m-1} = \begin{cases} \left\langle 2a + 1 - \frac{\ell}{2} \right\rangle, & \ell \text{ even;} \\ \left\langle 2a - \frac{\ell-1}{2} \right\rangle, & \ell \text{ odd;} \end{cases}$$

$$E_3^{m-\ell-1,m} = \begin{cases} \mathbb{Z}\,, & \ell = m - 1; \\ \left\langle 2a + 1 - \frac{\ell}{2} \right\rangle, & \ell \text{ even, } \ell < m - 1; \\ \left\langle 2a - \frac{\ell+1}{2} \right\rangle, & \ell \text{ odd;} \end{cases}$$

$$E_3^{2m-\ell,0} = \begin{cases} \left\langle 4a + 2 - \frac{\ell}{2} \right\rangle, & \ell \equiv 0 \bmod 4; \\ \left\{ 4a + 1 - \frac{\ell}{2} \right\} & \ell \equiv 2 \bmod 4; \\ \left\langle 4a - \frac{\ell-1}{2} \right\rangle, & \text{otherwise;} \end{cases}$$

and since $E_{m+2}^{2m-\ell,0}$ has 2-rank at most $r_{2m-\ell}$ (indeed, $E_{m+2}^{2m-\ell,0} = E_\infty^{2m-\ell,0}$ which is a subgroup of $H^{2m-\ell}(B(\mathrm{P}^m, 2))$), an easy counting argument (using, as in the case of an even m, the right exactness of the tensor product) gives that the differentials in (27) and (28) must yield an injective map after tensoring with $\mathbb{Z}_2$. In particular they

(a) must be injective on the nose, except for the case discussed in (29);

(b) cannot hit twice the generator of a $\mathbb{Z}_4$-summand.

The already observed equalities $E_2^{0,2m-1} = H^0(BD_8;\mathbb{Z}_\alpha) = 0$ together with (a) above imply that, in total dimensions t with $t \leq 2m-1$ and $t \neq m$, the E_{m+2}-term of the spectral sequence is concentrated on the base line ($q = 0$), while at higher lines ($q > 0$) the spectral sequence only has a $\mathbb{Z}$-group—at node $(0,m)$. This situation yields Theorem 1.8, while (b) above yields Proposition 1.9. □

A direct calculation (left to the reader) using the proved behavior of the differentials in (27) and (28)—and using (twice) the analogue of Proposition 5.2 when $\ell \equiv 2 \bmod 4$—gives

$$H^{2m-\ell}(B(\mathrm{P}^m,2)) = \begin{cases} \left\langle \frac{\ell}{2} \right\rangle, & \ell \equiv 0 \bmod 4; \\ \left\{ \frac{\ell}{2} - 1 \right\}, & \ell \equiv 2 \bmod 4; \\ \left\langle \frac{\ell+1}{2} \right\rangle, & \text{otherwise}; \end{cases}$$

for $2 \leq \ell \leq m-1$. Thus, as the reader can easily check using Corollaries 2.4 and 2.5, instead of the symmetry isomorphisms exemplified in Table 1, the cohomology groups of $B(\mathrm{P}^m,2)$ are now formed (as predicted by the isomorphisms (16) of the previous section) by a combination of $H^*(BD_8)$ and $H^*(BD_8;\mathbb{Z}_\alpha)$—in the lower and upper halves, respectively. Once again, the CLSS analysis not only offers an alternative to the (torsion linking form) arguments in the previous section, but it allows us to recover, under the present hypotheses, the torsion subgroup in the three missing dimensions in (17).

Example 5.7. For $m \equiv 1 \bmod 4$, [3, Theorem 1.4 (D)] identifies two explicit elements in the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m,2))$: one for each of $i = m+1$ and $i = m+3$. In particular, this produces at most three basis elements in the ideal $\mathrm{Ker}(p^*)$ in dimensions at most $m+3$. However it follows from the previous spectral sequence analysis that, for $m+1 \leq i \leq 2m-1$, the kernel of $p^*\colon H^i(BD_8) \to H^i(B(\mathrm{P}^m,2))$ is an $\mathbb{F}_2$-vector space of dimension $i - m + (-1)^i$. This means that through dimensions at most $m+3$ (and with $m \geq 5$) there are at least four more basis elements remaining to be identified in $\mathrm{Ker}(p^*)$.

6 Case of $B(\mathrm{P}^{4a+3},2)$

We now discuss some aspects of the spectral sequence of the previous section in the unresolved case $m \equiv 3 \bmod 4$. Although we are unable to describe the pattern of differentials for such m, we show that enough information can be collected to not only resolve the three missing cases

in (17), but also to conclude the proof of Theorem 1.8 for $G = D_8$. Unless explicitly stated otherwise, the hypothesis $m \equiv 3 \bmod 4$ will be in force throughout the section.

Remark 6.1. The main problem that has prevented us from fully understanding the spectral sequence of this section comes from the apparent fact that the algebraic input coming from the $H^*(BD_8)$-module structure in the CLSS—the crucial property used in the proof of Lemma 5.5—does not give us enough information in order to determine the pattern of d_2-differentials. New geometric insights seem to be needed instead. Although it might be tempting to conjecture the validity of Lemma 5.5 for $m \equiv 3 \bmod 4$, we have not found concrete evidence supporting such a possibility. In fact, a careful analysis of the possible behaviors of the spectral sequence for $m = 3$ (performed in Section 7) does not give even a more aesthetically pleasant reason for leaning toward the possibility of having a valid Lemma 5.5 in the current congruence. A second problem arose in [13] when we noted that, even if the pattern of d_2-differentials were known for $m \equiv 3 \bmod 4$, there would seem to be a slight indeterminacy either in a few higher differentials (if Lemma 5.5 holds for $m \equiv 3 \bmod 4$), or in a few possible extensions among the $E_\infty^{p,q}$ groups (if Lemma 5.5 actually fails for $m \equiv 3 \bmod 4$). Even though we cannot resolve the current d_2-related ambiguity, in [13, Example 6.4] we note that, at least for $m = 3$, it is possible to overcome the above mentioned problems about higher differentials or possible extensions by making use of the explicit description of $H^4(B(\mathrm{P}^3, 2))$—given later in the section (considerations previous to Remark 6.3) in regard to the claimed surjectivity of (18); see also [12], where advantage is taken of the fact that P^3 is a group. The explicit possibilities in the case of P^3 are discussed in Section 7.

In the first result of this section, Theorem 1.8 for $G = D_8$ and $m \equiv 3 \bmod 4$, we show that, despite the previous comments, the spectral sequence approach can still be used to compute $H^*(B(\mathrm{P}^{4a+3}, 2))$ just beyond the middle dimension (i.e., just before the first problematic d_2-differential plays a decisive role). In particular, this computes the corresponding groups in the first two of the three missing cases in (17).

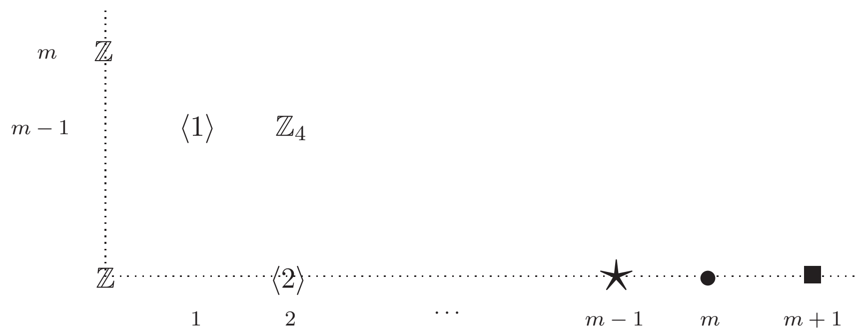
Proposition 6.2. *Let $m = 4a+3$. The map $H^i(BD_8) \to H^i(B(\mathrm{P}^m, 2))$ induced by* (2) *is:*

1. *an isomorphism for $i < m$;*

2. *a monomorphism onto the torsion subgroup of* $H^i(B(\mathrm{P}^m, 2)) = \langle 2a+1\rangle \oplus \mathbb{Z}$ *for* $i = m$;

3. *the zero map for* $2m - 1 < i$.

Proof. The argument parallels that used in the analysis of the CLSS when $m \equiv 1 \bmod 4$. Here is the chart of the current E_2-term through total dimensions at most $m + 1$:



The star at node $(m-1, 0)$ stands for $\langle 2a+2\rangle$; the bullet at node $(m, 0)$ stands for $\langle 2a+1\rangle$; the solid box at node $(m+1, 0)$ stands for $\{2a+2\}$. In this range there are only three possibly nonzero differentials:

- a d_2 from node $(0, m)$ to node $(2, m-1)$;
- a d_m from node $(1, m-1)$ to node $(m+1, 0)$;
- a d_{m+1} from node $(0, m)$ to node $(m+1, 0)$.

Whatever these d_2 and d_{m+1} are, there will be a resulting $E_\infty^{0,m} = \mathbb{Z}$. On the other hand, the argument about 2-ranks in (20) and in (30), leading respectively to (21) and (31), now yields that the torsion 2-group $H^{m+1}(B(\mathrm{P}^m, 2))$ has 2-rank $2a+1$. Since $E_\infty^{m+1,0}$ is a subgroup of $H^{m+1}(B(\mathrm{P}^m, 2))$, this forces the two differentials d_m and d_{m+1} above to be nonzero, each one with cokernel of 2-rank one less than the 2-rank of its codomain. In fact, d_m must have cokernel isomorphic to $\{2a+1\}$, whereas the cokernel of d_{m+1} is either $\{2a\}$ or $\langle 2a+1\rangle$ (Remark 6.3, and especially [13, Example 6.4], expand on these possibilities). What matters here is the forced injectivity of d_m, which implies $E_\infty^{1,m-1} = 0$ and, therefore, the second assertion of the proposition—the first assertion is obvious from the CLSS, while the third one is elementary. □

We now start work on the only groups in Theorem 1.2 not yet computed, namely $H^{m+1}(B(\mathrm{P}^m, 2))$ for $m = 4a + 3$. As indicated in the

previous proof, these are torsion 2-groups of 2-rank $2a+1$. Furthermore, (18) and Corollary 2.5 show that each such group contains a copy of $\{2a\}$, a 2-group of the same 2-rank as that of $H^{m+1}(B(\mathrm{P}^m,2))$. In showing that the two groups actually agree (thus completing the proof of Theorem 1.2), a key fact comes from Fred Cohen's observation (recalled in the paragraph previous to Remark 1.3) that *there are no elements of order* 8. For instance,

(32) when $m=3$ the two groups must agree since both are cyclic (i.e., have 2-rank 1).

In order to deal with the situation for positive values of a, Cohen's observation is coupled with a few computations in the first two pages of the Bockstein spectral sequence (BSS) for $B(\mathrm{P}^m,2)$: we will show that there is only one copy of $\mathbb{Z}_4$ (the one coming from the subgroup $\{2a\}$) in the decomposition of $H^{m+1}(B(\mathrm{P}^m,2))$ as a sum of cyclic 2-groups—forcing $H^{m+1}(B(\mathrm{P}^m,2))=\{2a\}$.

Remark 6.3. Before undertaking the BSS calculations (in Proposition 6.4 below), we pause to observe that, unlike the Bockstein input in all the previous CLSS-related proofs, the use of the BSS does not seem to give quite enough information in order to understand the pattern of d_2-differentials in the current CLSS. Much of the problem lies in being able to decide the actual cokernel of the d_{m+1}-differential in the previous proof and, consequently, understand how the $\mathbb{Z}_4$-group in $H^{m+1}(B(\mathrm{P}^m,2))$ arises in the current CLSS; either entirely at the $q=0$ line (as in all cases of the previous—and the next—section), or as a nontrivial extension in the E_∞ chart. The final section of the paper discusses in detail these possibilities in the case $m=3$—which should be compared to the much simpler situation in Example 5.1.

Recall from [8, 15] that the mod 2 cohomology ring of $B(\mathrm{P}^m,2)$ is polynomial on three classes x, x_1, and x_2, of respective dimensions 1, 1, and 2, subject to the three relations

(I) $x^2 = xx_1$;

(II) $$\sum_{0\le i\le \frac{m}{2}} \binom{m-i}{i} x_1^{m-2i}x_2^i = 0;$$

(III) $$\sum_{0\le i\le \frac{m+1}{2}} \binom{m+1-i}{i} x_1^{m+1-2i}x_2^i = 0.$$

Further, the action of Sq^1 is determined by (I) and

$$\mathrm{Sq}^1 x_2 = x_1 x_2. \tag{33}$$

[The following observations—proved in [8, 15], but not needed in this paper—might help the reader to assimilate the facts just described: The three generators x, x_1, and x_2 are in fact the images under the map p_{m,D_8} in (2) of the corresponding classes at the beginning of Section 2. In turn, the latter generators x_1 and x_2 come from the Stiefel-Whitney classes w_1 and w_2 in $B\mathrm{O}(2)$ under the classifying map for the inclusion $D_8 \subset \mathrm{O}(2)$. In these terms, (33) corresponds to the (simplified in $B\mathrm{O}(2)$) Wu formula $\mathrm{Sq}^1(w_2) = w_1 w_2$. Finally, the two relations (II) and (III) correspond to the fact that the two dual Stiefel-Whitney classes $\overline{w}_m$ and $\overline{w}_{m+1}$ in $B\mathrm{O}(2)$ generate the kernel of the map induced by the Grassmann inclusion $G_{m+1,2} \subset B\mathrm{O}(2)$.]

Let R stand for the subring generated by x_1 and x_2, so that there is an additive splitting

$$H^*(B(\mathrm{P}^m, 2); \mathbb{F}_2) = R \oplus x \cdot R \tag{34}$$

which is compatible with the action of Sq^1 (note that multiplication by x determines an additive isomorphism $R \cong x \cdot R$).

Proposition 6.4. *Let $m = 4a+3$. With respect to the differential Sq^1 :*

- $H^{m+1}(R; \mathrm{Sq}^1) = \mathbb{Z}_2$.
- $H^{m+1}(x \cdot R; \mathrm{Sq}^1) = 0$.

Before proving this result, let us indicate how it can be used to show that (18) is an isomorphism for $m = 4a+3$. As explained in the paragraph containing (32), we must have

$$2 \cdot H^{4a+4}(B(\mathrm{P}^{4a+3}, 2)) = \langle r \rangle \quad \text{with} \quad r \geq 1 \tag{35}$$

and we need to show that $r = 1$ is in fact the case. Consider the Bockstein exact couple

$$\begin{tikzcd}
H^*(B(P^{4a+3},2)) \arrow[rr, "2"] & & H^*(B(P^{4a+3},2)) \arrow[dl, "\rho"] \\
& H^*(B(P^{4a+3},2);\mathbb{F}_2). \arrow[ul, "\delta"] &
\end{tikzcd}$$

In the (unravelled) derived exact couple

$$\cdots \to 2\cdot H^{4a+4}(B(P^{4a+3},2)) \xrightarrow{2} 2\cdot H^{4a+4}(B(P^{4a+3},2)) \to$$
$$\to H^{4a+4}(H^*(B(P^{4a+3},2);\mathbb{F}_2);\mathrm{Sq}^1) \to 2\cdot H^{4a+5}(B(P^{4a+3},2)) \to \cdots$$

we have $2\cdot H^{4a+5}(B(P^{4a+3},2)) = 0$ since $H^{4a+5}(B(P^{4a+3},2)) = \langle 2a+1\rangle$—argued in Section 4 by means of the (twisted) torsion linking form. Together with (35), this implies that the map

$$\text{(36)}\quad \langle r\rangle = 2\cdot H^{4a+4}(B(P^{4a+3},2)) \to H^{4a+4}(H^*(B(P^{4a+3},2);\mathbb{F}_2);\mathrm{Sq}^1)$$

in the above exact sequence is an isomorphism. Proposition 6.4 and (34) then imply the required conclusion $r = 1$.

Proof of Proposition 6.4. Note that every binomial coefficient in (II) with $i \not\equiv 0 \bmod 4$ is congruent to zero mod 2. Therefore relation (II) can be rewritten as

$$\text{(37)}\qquad x_1^{4a+3} = \sum_{j=1}^{a/2} \binom{a-j}{j} x_1^{4(a-2j)+3} x_2^{4j}.$$

Likewise, every binomial coefficient in (III) with $i \equiv 3 \bmod 4$ is congruent to zero mod 2. Then, taking into account (37), relation (III) becomes

$$\text{(38)}\quad \begin{aligned} x_2^{2a+2} &= x_1^{4a+4} + \sum_{i\in\Lambda} \binom{4a+4-i}{i} x_1^{4a+4-2i} x_2^{i} \\ &= \sum_{j=1}^{a/2} \binom{a-j}{j} x_1^{4(a-2j)+4} x_2^{4j} + \sum_{i\in\Lambda} \binom{4a+4-i}{i} x_1^{4a+4-2i} x_2^{i} \end{aligned}$$

where Λ is the set of integers i with $1 \le i \le 2a+1$ and $i \not\equiv 3 \bmod 4$. Using (37) and (38) it is a simple matter to write down a basis for R and $x\cdot R$ in dimensions $4a+3$, $4a+4$, and $4a+5$. The information is summarized (under the assumption $a > 0$, which is no real restriction in view of (32)) in the following chart, where elements in a column form a basis in the indicated dimension, and where crossed out terms can be expressed as linear combination of the other ones in view of (37) and (38).

$$
\begin{array}{lclcl}
4a+3 & & 4a+4 & & 4a+5 \\
\cancel{x_1^{4a+3}} & & \cancel{x_1^{4a+4}} & & \cancel{x_1^{4a+5}} \\
x_1^{4a+1}x_2 & \longrightarrow 0 & x_1^{4a+2}x_2 & & \cancel{x_1^{4a+3}x_2} \\
x_1^{4a-1}x_2^2 & \longrightarrow & x_1^{4a}x_2^2 & & x_1^{4a+1}x_2^2 \\
x_1^{4a-3}x_2^3 & \longrightarrow 0 & x_1^{4a-2}x_2^3 & \longrightarrow & x_1^{4a-1}x_2^3 \\
\vdots & & \vdots & & x_1^{4a-3}x_2^4 \\
x_1^3x_2^{2a} & \longrightarrow & x_1^4x_2^{2a} & & \vdots \\
x_1x_2^{2a+1} & \longrightarrow 0 & x_1^2x_2^{2a+1} & \longrightarrow & x_1^3x_2^{2a+1} \\
 & & \cancel{x_2^{2a+2}} & & \cancel{x_1x_2^{2a+2}} \\
\hline
xx_1^{4a+2} & & \cancel{xx_1^{4a+3}} & & \cancel{xx_1^{4a+4}} \\
xx_1^{4a}x_2 & \longrightarrow 0 & xx_1^{4a+1}x_2 & \longrightarrow & xx_1^{4a+2}x_2 \\
xx_1^{4a-2}x_2^2 & \longrightarrow & xx_1^{4a-1}x_2^2 & & xx_1^{4a}x_2^2 \\
\vdots & & xx_1^{4a-3}x_2^3 & \longrightarrow & xx_1^{4a-2}x_2^3 \\
 & & \vdots & & \vdots \\
xx_1^2x_2^{2a} & \longrightarrow & xx_1^3x_2^{2a} & & \\
xx_2^{2a+1} & \longrightarrow 0 & xx_1x_2^{2a+1} & \longrightarrow & xx_1^2x_2^{2a+1} \\
 & & & & \cancel{xx_2^{2a+2}}
\end{array}
$$

The top and bottom portions of the chart (delimited by the horizontal dotted line) correspond to R and $x \cdot R$, respectively. Horizontal arrows indicate Sq^1-images, which are easily computable from (33) and (I):

$$\mathrm{Sq}^1(x^i x_1^{i_1} x_2^{i_2}) = 0$$

when $i + i_1 + i_2$ is even, while

$$\mathrm{Sq}^1(x^i x_1^{i_1} x_2^{i_2}) = x^i x_1^{i_1+1} x_2^{i_2}$$

when $i + i_1 + i_2$ is odd—here $i \in \{0, 1\}$ in view of (I) above. There are only two basis elements, in dimensions $4a + 3$ and $4a + 4$, whose Sq^1-images are not indicated in the chart: $xx_1^{4a+2} \in (x \cdot R)^{4a+3}$ and $x_1^{4a+2}x_2 \in R^{4a+4}$. The second conclusion in the proposition is evident from the bottom part of the chart—no matter what the Sq^1-image of xx_1^{4a+2} is. On the other hand, the top portion of the chart implies that, in dimension $4a + 4$, $\mathrm{Ker}(\mathrm{Sq}^1)$ and $\mathrm{Im}(\mathrm{Sq}^1)$ are elementary 2-groups whose ranks satisfy

$$\mathrm{rk}(\mathrm{Ker}(\mathrm{Sq}^1)) = \mathrm{rk}(\mathrm{Im}(\mathrm{Sq}^1)) + \varepsilon$$

with $\varepsilon = 1$ or $\varepsilon = 0$ (depending on whether or not $\mathrm{Sq}^1(x_1^{4a+2}x_2)$ can be written down as a linear combination of the elements $x_1^{4a-1}x_2^3$, $x_1^{4a-5}x_2^5, \ldots,$ and $x_1^3x_2^{2a+1}$—this of course depends on the actual binomial coefficients in (37)). But the possibility $\varepsilon = 0$ is ruled out by (35) and (36), forcing $\varepsilon = 1$ and, therefore, the first assertion of this proposition. $\square$

7 The CLSS for $B(\mathrm{P}^3, 2)$

Here is the chart for the E_2-term of the spectral sequence for $m = 3$ through filtration degree 13:

$$
\begin{array}{c|cccccccccccccccc}
5 & & \langle 1\rangle & \{0\} & \langle 2\rangle & \langle 2\rangle & \langle 3\rangle & \{2\} & \langle 4\rangle & \langle 4\rangle & \langle 5\rangle & \{4\} & \langle 6\rangle & \langle 6\rangle & \langle 7\rangle & \cdots \\
4 & & & & & & & & & & & & & & & \\
3 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \{2\} & \langle 2\rangle & \langle 4\rangle & \langle 3\rangle & \{4\} & \langle 4\rangle & \langle 6\rangle & \langle 5\rangle & \{6\} & \langle 6\rangle & \cdots \\
2 & & \langle 1\rangle & \{0\} & \langle 2\rangle & \langle 2\rangle & \langle 3\rangle & \{2\} & \langle 4\rangle & \langle 4\rangle & \langle 5\rangle & \{4\} & \langle 6\rangle & \langle 6\rangle & \langle 7\rangle & \cdots \\
1 & & & & & & & & & & & & & & & \\
0 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \{2\} & \langle 2\rangle & \langle 4\rangle & \langle 3\rangle & \{4\} & \langle 4\rangle & \langle 6\rangle & \langle 5\rangle & \{6\} & \langle 6\rangle & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 &
\end{array}
$$

Since $H^5(B(\mathrm{P}^3, 2)) = \mathbb{Z}_2$ (Corollary 3.2), there must be a nontrivial differential landing at node $(5, 0)$. The only such possibility is

$$d_3^{2,2}\colon E_3^{2,2} = \mathbb{Z}_4 \Big/ \mathrm{Im}(d_2^{0,3}) \to E_3^{5,0} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \tag{39}$$

which, up to a change of basis, is the composition of the canonical projection $\mathbb{Z}_4 \Big/ \mathrm{Im}(d_2^{0,3}) \to \mathbb{Z}_2$ and the canonical inclusion $\iota_1\colon \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In particular, as in the conclusion of the second paragraph of the proof of Lemma 5.5, the differential $d_2^{0,3}\colon E_2^{0,3} = \mathbb{Z} \to E_2^{2,2} = \mathbb{Z}_4$ cannot be surjective (otherwise (39) would be the zero map) and, therefore, its only options are:

(40) $d_2^{0,3}$ is trivial, or

(41) as in (23), $d_2^{0,3}$ is twice the canonical projection.

The goal in this example is to discuss how neither of these two options leads to an apparent contradiction in the behavior of the spectral sequence. As a first task we consider the situation where (40) holds, noticing that if $d_2^{0,3}$ vanishes, then the $H^*(BD_8)$-module structure in the spectral sequence implies that the whole $(q = 3)$-line consists of d_2-cycles, so the above chart actually gives the E_3-term. Furthermore, using again the $H^*(BD_8)$-module structure, we note that every d_3-differential from the $(q = 2)$-line to the $(q = 0)$-line would have to repeat vertically as a d_3-differential from the $(q = 5)$-line to the $(q = 3)$-line.

Under these conditions, let us now analyze d_3-differentials. The proof of Proposition 6.2 already discusses the d_3-differential (and its cokernel) from node $(1,2)$ to node $(4,0)$. On the other hand, the d_3-differential from node $(2,2)$ to node $(5,0)$ is (39) and has been fully described. Note that the behavior of these two initial d_3-differentials can be summarized by remarking that they yield monomorphisms after tensoring with $\mathbb{Z}_2$. We now show, by means of a repeated cycle of three steps, that this is also the case for all the remaining d_3-differentials.

Step 1. To begin with, observe that the argument in the final paragraph of the proof of Lemma 5.5 does not lead to a contradiction: it only implies that both differentials $d_3\colon E_3^{3,2} \to E_3^{6,0}$ and $d_4\colon E_4^{2,3} \to E_4^{6,0}$ must be injective—this time wiping out $E_\infty^{2,3}$, $E_\infty^{3,2}$, and $E_\infty^{6,0}$.

Step 2. In view of our discussion of the first nontrivial d_3-differential, the last assertion in the paragraph following (41) implies that the group $\langle 1\rangle$ at node $(1,5)$ does not survive to E_4; indeed, the differential

$$d_3\colon E_3^{1,5} = \langle 1\rangle \to E_3^{4,3} = \{2\}$$

is injective with cokernel $E_4^{4,3} = \{1\}$. Such a situation has two consequences. First, that the discussion in the previous step applies word for word when the three nodes $(2,3)$, $(3,2)$, and $(6,0)$ are respectively replaced by $(3,3)$, $(4,2)$, and $(7,0)$. Second, that there is no room for a nonzero differential landing in $E_i^{5,2}$ or $E_j^{4,3}$ for $i \geq 3$ and $j \geq 4$ (of course we have detected the nontrivial differential d_3 landing at node $(4,3)$), so that both $d_3^{5,2}$ and $d_4^{4,3}$ must be injective (recall $H^7(B(\mathrm{P}^3,2)) = 0$). Actually, the only way for this to (algebraically) hold is with an injective $d_3^{5,2} \otimes \mathbb{Z}_2$.

Step 3. Note that the differential $d_3^{6,2}\colon E_3^{6,2} = \{2\} \to E_3^{9,0} = \langle 4\rangle$ has at least a $\mathbb{Z}_2$-group in its kernel. But the kernel cannot be any larger: the only nontrivial differential landing at node $(6,2)$ starts at node $(2,5)$ and, as we already showed, $E_4^{2,5} = \mathbb{Z}_2$. Consequently, $d_3^{6,2} \otimes \mathbb{Z}_2$ is injective.

The arguments in these three steps repeat, essentially word for word, in a periodic way, each time accounting for the $(-\otimes\mathbb{Z}_2)$-injectivity of the next block of four consecutive d_3-differentials. This leads to the following chart of the resulting E_4-term (again through filtration degree 13):

$$
\begin{array}{c|cccccccccccccccc}
5 & & & \langle 1\rangle & & & & \langle 1\rangle & & & & \langle 1\rangle & & & & \cdots \\
4 & & & & & & & & & & & & & & & \\
3 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \cdots \\
2 & & & \langle 1\rangle & & & & \langle 1\rangle & & & & \langle 1\rangle & & & & \cdots \\
1 & & & & & & & & & & & & & & & \\
0 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \langle 2\rangle & \langle 1\rangle & \{1\} & \langle 1\rangle & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 &
\end{array}
$$

At this point further differentials are forced just from the fact that $H^i(B(\mathrm{P}^3, 2)) = 0$ for $i \geq 6$. Indeed, all possibly nontrivial differentials $d_4^{p,q}$ must be isomorphisms for $p \geq 2$, whereas the $H^*(BD_8)$-module structure implies that the image of the differential $d_4^{0,3}\colon E_4^{0,3} = \mathbb{Z} \to E_4^{4,0} = \{1\}$ is generated by an element of order 4. Thus, the whole E_5-term reduces to the chart:

$$
\begin{array}{c|cccccc}
3 & \mathbb{Z} & & & & & \\
2 & & & \langle 1\rangle & & & \\
1 & & & & & & \\
0 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

This is also the E_∞-term for dimensional reasons, and the resulting output is compatible with the known structure of $H^*(B(\mathrm{P}^3, 2))$—note that the only possibly nontrivial extension (in total degree 4) is actually nontrivial, in view of [12, Theorem 1.5]. This concludes our discussion of the first task in this section, namely, that (40) leads to no apparent contradiction in the behavior of the spectral sequence (alternatively: the breakdown in the proof of Lemma 5.5 for $m = 3$, already observed in Step 1 above, does not seem to be fixable with the present methods).

The second and final task in this section is to explain how, just as (40) does, option (41) leads to no apparent contradiction in the behavior of the spectral sequence. Thus, for the remainder of the section we assume (41). In particular, the $H^*(BD_8)$-module structure in the spectral sequence implies that the conclusion of Lemma 5.5 holds. Then, as explained in the paragraph following Remark 5.6, the resulting E_3-

term now takes the form

$$
\begin{array}{c|cccccccccccccc}
5 & & \langle 1\rangle & \{0\} & \langle 2\rangle & \langle 2\rangle & \langle 3\rangle & \{2\} & \langle 4\rangle & \langle 4\rangle & \langle 5\rangle & \{4\} & \langle 6\rangle & \langle 6\rangle & \langle 7\rangle \ \cdots \\
4 & & & & & & & & & & & & & & \\
3 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \langle 3\rangle & \langle 2\rangle & \langle 4\rangle & \langle 3\rangle & \langle 5\rangle & \langle 4\rangle & \langle 6\rangle & \langle 5\rangle & \langle 7\rangle & \langle 6\rangle \ \cdots \\
2 & & \langle 1\rangle & \langle 1\rangle & \langle 2\rangle & \langle 2\rangle & \langle 3\rangle & \langle 3\rangle & \langle 4\rangle & \langle 4\rangle & \langle 5\rangle & \langle 5\rangle & \langle 6\rangle & \langle 6\rangle & \langle 7\rangle \ \cdots \\
1 & & & & & & & & & & & & & & \\
0 & \mathbb{Z} & & \langle 2\rangle & \langle 1\rangle & \{2\} & \langle 2\rangle & \langle 4\rangle & \langle 3\rangle & \{4\} & \langle 4\rangle & \langle 6\rangle & \langle 5\rangle & \{6\} & \langle 6\rangle \ \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\end{array}
$$

where again only dimensions at most 13 are shown.

At this point it is convenient to observe that the last statement in the paragraph following (41) fails under the current hypothesis. Indeed, the generator of $E_3^{0,3}$ is twice the generator of $E_2^{0,3}$, breaking up the vertical symmetry of d_3-differentials holding under (40)—of course, the groups in the current E_3-term already lack the vertical symmetry we had in the case of (40). In order to deal with such an asymmetric situation we need to make a *differential-wise* measurement of all the groups involved in the current E_3-term (we will simultaneously analyze the possibilities for the two horizontal families of d_3-differentials).

To begin with, note that the arguments dealing, in the case of (40), with the two differentials $E_3^{1,2} \to E_3^{4,0}$ and $E_3^{2,2} \to E_3^{5,0}$ apply without change under the current hypothesis to yield that these two differentials are injective, the former with cokernel $E_4^{4,0} = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ (i.e., both yield injective maps after tensoring with $\mathbb{Z}_2$). Note that any other group not appearing as the domain or codomain of these two differentials must be eventually wiped out in the spectral sequence, either because $H^i(B(\mathrm{P}^3, 2)) = 0$ for $i \geq 6$, or else because the already observed $E_4^{5,0} = \mathbb{Z}_2$ accounts for all there is in $H^5(B(\mathrm{P}^3, 2))$ in view of Corollary 3.2. This observation is the key in the analysis of further differentials, which uses repeatedly the following three-step argument (the reader is advised to keep handy the previous chart in order to follow the details):

Step 1. The groups $E_3^{p,q}$ not yet considered and having smallest $p+q$ are $E_3^{3,2}$ and $E_3^{2,3}$. Both are isomorphic to $\langle 2\rangle$; none can be hit a differential. Since $E_3^{6,0} = \langle 4\rangle$, we must have injective differentials $d_3\colon E_3^{3,2} \to E_3^{6,0}$ and $d_4\colon E_4^{2,3} \to E_4^{6,0}$, clearing the E_∞-term at nodes $(2,3)$, $(3,2)$, and

$(6,0)$. Look now at the groups not yet considered and in the next smallest total dimension $p+q$. These are $E_3^{1,5}$, $E_3^{3,3} = \langle 1\rangle$, and $E_3^{4,2} = \langle 2\rangle$. Again the last two cannot be hit by a differential and, since $E_3^{7,0} = \langle 3\rangle$, the two differentials $d_3\colon E_3^{4,2} \to E_3^{7,0}$ and $d_4\colon E_4^{3,3} \to E_4^{7,0}$ must be injective, now clearing the E_∞-term at nodes $(3,3)$, $(4,2)$, and $(7,0)$.

Step 2. The only case remaining to consider with $p+q = 5$ is $E_3^{1,5} = \langle 1\rangle$. We have seen that there is nothing left in the spectral sequence for this group to hit with a d_6-differential, so it must hit either $E_3^{4,3} = \langle 3\rangle$ or $E_3^{5,2} = \langle 3\rangle$. Therefore, in these two positions there are 2^5 elements that will have to inject into (quotients of) $E_3^{8,0} = \{4\}$, a group with cardinality 2^6. The outcome of this situation is two-fold:

(i) the E_∞-term is now cleared at positions $(1,5)$, $(4,3)$, and $(5,2)$;

(ii) there is a $\mathbb{Z}_2$ group at node $(8,0)$ that still needs a differential matchup.

But (i) implies that the only way to kill the element in (ii) is with a d_6-differential originating at node $(2,5)$, where we have $E_3^{2,5} = \mathbb{Z}_4$.

Step 3. The above analysis leaves only one element at node $(2,5)$ still without a differential matchup. Since everything at node $(8,0)$ has been accounted for, the element in question at node $(2,5)$ must be cleared up at either of the stages E_3 or E_4 with a corresponding nontrivial differential landing at nodes $(5,3)$ or $(6,2)$, respectively. But $E_3^{5,3} = \langle 2\rangle$ while $E_3^{6,2} = \langle 3\rangle$. Thus, the last differential will *leave* 2^4 elements that need to be mapped injectively by *previous* differentials landing at node $(9,0)$. Since $E_3^{9,0} = \langle 4\rangle$, our bookkeeping analysis has now cleared up every group $E_\infty^{p,q}$ with either

- $q = 0$ and $p \leq 9$;
- $q = 2$ and $p \leq 6$;
- $q = 3$ and $p \leq 5$;
- $q = 5$ and $p \leq 2$.

These three steps now repeat to cover the next four cases of p. For instance, one starts by looking at $E_3^{3,5} = \langle 2\rangle$, whose two basis elements are forced to inject with differentials landing either at node $(6,3)$ or $(7,2)$. Since $E_3^{6,3} \cong E_3^{7,2} \cong \langle 4\rangle$, this leaves 2^6 elements that must be mapping into node $(10,0)$ through injective differentials. But $E_3^{10,0} =$

$\langle 6 \rangle$, clearing the appropriate nodes—the situation in Step 1. At the end of this three-step inductive analysis we find that there is just the right number of elements, at the right nodes, to match up through differentials—the opposite of the situation that we successfully exploited in the previous section to deal with cases where $m \not\equiv 3 \bmod 4$.

From the chart we note that $d_4\colon E_4^{0,3} = \mathbb{Z} \to E_4^{4,0} = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is the only undecided differential, and that its cokernel equals $H^4(B(\mathrm{P}^3, 2))$—since $E_\infty^{2,2} = 0 = E_\infty^{1,3}$. The two possibilities (indicated at the end of the proof of Proposition 6.2) for this cokernel are $\mathbb{Z}_2$ and $\mathbb{Z}_4$, but [12, Theorem 1.5] implies that the latter option must be the right one under the present hypothesis (41).

Remark 7.1. The previous paragraph suggests that, if our methods are to be used to understand the CLSS in the remaining case with $m \equiv 3 \bmod 4$, then it will be convenient to keep in mind the type of 2^e-torsion Theorem 1.2 describes for the integral cohomology of $B(\mathrm{P}^{4a+3}, 2)$.

8 Case of $F(\mathrm{P}^m, 2)$

The CLSS analysis in the previous two sections can be applied—with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $G = D_8$—in order to study the cohomology groups of the ordered configuration space $F(\mathrm{P}^m, 2)$. The explicit details are similar but much easier than those for unordered configuration spaces, and this time the additive structure of differentials can be fully understood for any m. Here we only review the main differences, simplifications, and results.

For one, there is no 4-torsion to deal with (e.g. the arithmetic Proposition 5.2 is not needed); indeed, the role of BD_8 in the situation of an unordered configuration space $B(\mathrm{P}^m, 2)$ is played by $\mathrm{P}^\infty \times \mathrm{P}^\infty$ for ordered configuration spaces $F(\mathrm{P}^m, 2)$. Thus, the use of Corollaries 2.4 and 2.5 is replaced by the simpler Lemma 2.8. But the most important simplification in the calculations relevant to the present section comes from the absence of problematic d_2-differentials, the obstacle that prevented us from computing the CLSS of the D_8-action on $V_{m+1,2}$ for $m \equiv 3 \bmod 4$. [This is why in Lemma 2.8 we do not insist on describing $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty; \mathbb{Z}_\alpha)$ as a module over $H^*(\mathrm{P}^\infty \times \mathrm{P}^\infty)$—compare to Remark 5.6.] As a result, the integral cohomology CLSS of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action on $V_{m+1,2}$ can be fully understood, without restriction on m,

by means of the counting arguments used in Section 5, now forcing the injectivity of all relevant differentials from the following two ingredients:

(a) The size and distribution of the groups in the CLSS.

(b) The $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of Proposition 3.2 in Remark 3.3—the input triggering the determination of differentials.

In particular, when m is odd, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of Lemma 5.5 does not arise and, instead, only the counting argument in the proof following Remark 5.6 is needed.

We leave it for the reader to supply details of the above CLSS and verify that this leads to Propositions 1.7 and 1.8 in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, as well as to the computation of all the cohomology groups in Theorem 1.1.

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